PDE Observer Design for Counter-Current Heat Flows in a Heat-Exchanger

Fairouz Zobiri * **Emmanuel Witrant** * **François Bonne** **

* *Univ. Grenoble Alpes, GIPSA-lab, Grenoble, France, (e-mail: [fairouz.zobiri, emmanuel.witrant]@gipsa-lab.fr).*
** *UMR-E 9004 CEA/Univ. Grenoble Alpes, INAC, SBT, 17 rue des Martyrs, 38054 Grenoble, France, (e-mail: francois.bonne@cea.fr)*

Abstract: In this work, we consider the estimation of temperature profiles along the pipes of a plate heat exchanger. The transport phenomena through the heat exchanger are modeled by hyperbolic partial differential equations (PDE) of first order in time and space. The counter-flow heat exchange implies that the system is comprised of rightward (where the hot fluid circulates) and leftward (cold fluid pipes) hyperbolic PDE. The heat exchanged between the pipes of hot and cold fluid induces a coupling between the rightward and leftward equations, which increases the difficulty of solving the PDE system. The estimation objective is addressed by the design of an observer using a PDE approach, which uses boundary measurements to estimate the distributed profiles. The convergence of the observation error is established using Lyapunov analysis. Simulation results illustrate the efficiency of our method using a simulator with time-varying parameters validated on experimental data.

Keywords: Observers, partial differential equations, cryogenic temperatures, heat exchangers, distributed parameters.

1. INTRODUCTION

Heat exchangers are widely used and are present in a broad range of industrial applications: from the food and pharmaceutical industry to chemical plants and oil and gas refineries, they represent the tool of choice for cooling and heating operations. Though there exist several types of heat exchangers, the plate heat exchanger (PHE) is one of the most extensively employed type due to its high heat transfer efficiency in a compact size, combined with the ease of its maintenance and cleaning. Consequently, the control of such units becomes more and more important, especially as industries nowadays have increased the demand for higher heat transfer efficiency and lower energy consumption, as stated by Fratczak et al. (2014). However, the control and supervision of PHEs is a challenging task due to the time-varying and distributed nature of their transport parameters, requiring a full knowledge of the state while only boundary measurements are available. That is why, in this work, we apply a simplified model that captures all the dynamics necessary to the design of a good observer, while neglecting the effects that have little or no impact on the control problem. This work is motivated by and applied to an experimental helium refrigerator facility (available at CEA-INAC-SBT [1], Grenoble, France). We estimate the temperature profile along the plates of the heat exchanger. For this purpose, we build a boundary observer that reconstructs the temperature through the length of the exchanger, by only measuring temperature at the extremities of the system.

The transport and exchange of heat in the hot and cold fluids circulating in the heat exchanger are modeled with first order hyperbolic PDEs from the fundamental laws of physics (e.g. by Bonne et al., 2014). The classical engineering approach is to discretize the PDE and apply existing finite-dimensional control methods. However, some dynamics of the original (distributed-parameter) model are lost. This has led to the idea of extending the tools of classical control theory to encompass infinite-dimensional systems described by PDEs, without requiring the prior discretization of the PDE, thus preventing the loss of precious information on the transient behavior of the system. The stability of systems of first order hyperbolic PDEs has been proved by Rauch and Taylor (1975) using the method of characteristics and by Xu and Sallet (2002) with a Lyapunov function. Xu and Sallet (2002) also proved the exponential stability of heat exchanger networks.

Hyperbolic systems have been the subject of extensive research by the control community, due to their ubiquity in the physical world and the existence of various tools that facilitate the derivation of analytical solutions, such as the Riemann-invariants. Alinhac (2009) has, for example, dedicated an entire volume to the study of this class of PDE. Li (2010), on the other hand, proposes a thorough investigation of the controllability and observability of quasilinear hyperbolic systems. The issue of boundary control of such class of systems is also present in the literature. In Castillo et al. (2012) the problem of the stability of a boundary control applied to a system of one-dimensional rightward conservation transport PDE

⋆ This work has been partially supported by the French ANR project CryoGreen.

[1] Commissariat à l'Energie Atomique et aux Energies Alternatives - Institut NAnosciences et Cryogénie - Service des Basses Températures

is considered. Diagne et al. (2012) give a method for selecting the dissipative boundary conditions that ensure the stability of one-dimensional linear hyperbolic systems of balance laws, using Lyapunov exponential stability techniques. They use a strict Lyapunov approach to define the stabilizing boundary conditions for a 1-D hyperbolic PDE of conservation law and apply their results to the linearized Saint-Venant-Exner equations.

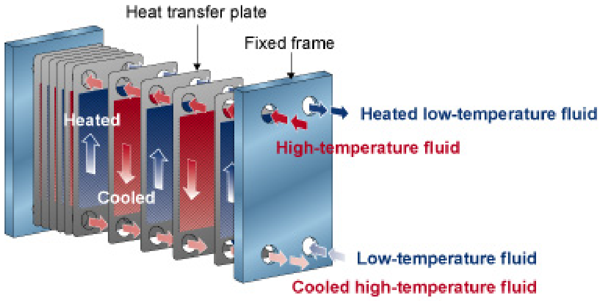
The question of observers for infinite-dimensional systems has been, however, less investigated. For systems of this type, where parameters are distributed, it is impossible to obtain measurements at each position in space. It is, therefore, more common for sensors to be located at the boundaries of the systems. Hence, the idea of boundary observers has been thriving. Krstic and Smyshlyaev (2008) use the backstepping method to design an observer for a class of parabolic PDEs. A boundary controller-observer is designed by Vazquez et al. (2011) for a 2×2 coupled linear hyperbolic system with spatially varying coefficients using a backstepping approach. In this work one boundary condition is supposed to be algebraically related to the other, which allows the authors to build an observer using the measurements from one end of the domain only. Similar ideas are used by Di Meglio et al. (2013) to develop a boundary controller-observer for a system of n-rightward and one leftward convecting transport PDE. Castillo et al. (2013) designed a boundary observer for a set of rightward quasi-linear hyperbolic transport equations of conservation laws, using the strict Lyapunov method developed by Coron et al. (2007). In this framework, our contribution is to synthesize a boundary observer for a system of linear, hyperbolic balance laws with a rightward and a leftward transport equation. For this, we get the inspiration from Castillo et al. (2012) for a type of infinite-dimensional Luenberger-type observer, which results in a (relatively) simple observer architecture, e.g. in comparison with the backstepping approaches. Then, we use the Lyapunov function proposed by Diagne et al. (2012) to determine the sufficient conditions for the convergence of the observer. Based on the Linear Matrix Inequalities obtained through this approach, we derive our observer gains.

This paper is organized as follows. The problem description is given in Section 2, providing the reference PDE model. The observer design is presented in Section 3, where we derive the specific conditions that ensure the exponential convergence of the observation error to zero. Section 4 is dedicated to the presentation of the simulation results for different cases that allow us to assess the performance and robustness of the observer.

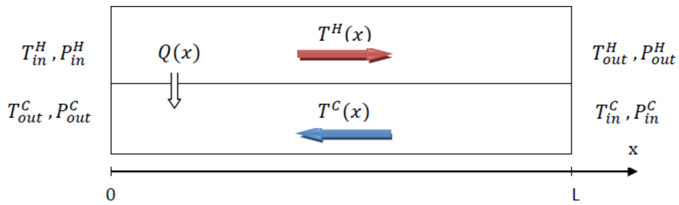
2. PROBLEM DESCRIPTION

The aim of this section is to provide the mathematical model of heat transport in a plate heat exchanger and to formulate the observation problem.

PHEs, such as the one represented in Fig. 1(a), consist of a set of plates arranged in a parallel configuration, forming channels through which hot and cold fluids (helium, in our case) flow. The PHE we study in this work is a counter-flow heat exchanger, in which the hot and cold fluids flow in opposite directions, thus maximizing heat transfer. The hot fluid enters the PHE from the top and exits at the bottom whereas the cold fluid follows the reverse path. The PHE is equipped with gaskets that prevent the two fluids from mixing, implying that the hot fluid flows through the even-numbered channels while the cold fluid moves inside the odd-numbered ones. In this work, we consider the simplified model of the PHE shown in Fig. 1(b), where the heat flow in each plate is supposed to be homogeneous and the heat exchange can be modeled in a single dimension. $T(x,t)$ and $Q(x,t)$ denote the distributed temperature and heat exchange, respectively, at location x and time t, the subscripts in/out denote inflow and outflow directions and the superscripts H and C refer to the hot and cold channels.



(a) Inside view (http://www.kobelco.co.jp/english/about_kobelco/csr/environment/2012/17.html).



(b) Simplified model.

Fig. 1. Plate heat exchanger.

The mathematical model is derived based on the following assumptions:

- we consider a one-dimensional flow along the x-axis (positive in the direction of the flow of the hot fluid);
- the heat diffusion phenomena through the plates of the exchanger have almost no impact on the overall behavior of the system and are neglected;
- the only categories of heat transfers are convection inside each flow and heat exchange between the flows;
- the heat exchanger is perfectly isolated and does not exchange heat with its surrounding environment;
- the heat transfer coefficient is supposed to be constant.

The PHE can then be described with the system of first-order hyperbolic partial differential equations provided by Bonne et al. (2014), $\forall x \in [0,\, 1]$:

$$\begin{cases} \dfrac{\partial T^H}{\partial t} = -\dfrac{F_1}{E_1}\dfrac{\partial T^H}{\partial x} - \dfrac{h}{E_1}(T^H - T^C) \\ \dfrac{\partial T^C}{\partial t} = \dfrac{F_2}{E_2}\dfrac{\partial T^C}{\partial x} + \dfrac{h}{E_2}(T^H - T^C) \end{cases} \tag{1}$$

with (in normalized space):

$$F_1 = \dot{M}^H C_p^H, \quad E_1 = \rho^H C_p^H V^H + \frac{M^{Al} C_p^{Al}}{2}$$
$$F_2 = \dot{M}^C C_p^C, \quad E_2 = \rho^C C_p^C V^C + \frac{M^{Al} C_p^{Al}}{2}$$

where $\dot{M}$ is the mass flow rate, ρ the density, C_p the specific heat, V the volume occupied by the fluid, h the heat transfer coefficient, M the PHC mass. The superscript Al denotes the Aluminum material. This system has boundary conditions of the form:

$$T^H(0,t) = T_{in}^H(t), \quad T^C(1,t) = T_{in}^C(t) \tag{2}$$

and initial conditions:

$$T^H(x,0) = T_0^H(x), \quad T^C(x,0) = T_0^C(x) \tag{3}$$

Considering the vector of temperatures $T = [T^H \, T^C]^T$, the system of PDEs (1) can be written in the state-space form, $\forall x \in [0, 1]$:

$$\partial_t T + A \partial_x T = BT \tag{4}$$

with:

$$A = \begin{bmatrix} \frac{F_1}{E_1} & 0 \\ 0 & -\frac{F_2}{E_2} \end{bmatrix}, \; B = \begin{bmatrix} -\frac{h}{E_1} & \frac{h}{E_1} \\ \frac{h}{E_2} & -\frac{h}{E_2} \end{bmatrix},$$

where we used the abbreviated notations $\partial_t = \partial/\partial t$ and $\partial_x = \partial/\partial x$.

In order to use the temperature along the channels of the heat exchanger for control and/or supervision purposes, we need to have access to the values of these temperatures at every time instant and at each point in space. Nevertheless, due to the costliness and infeasibility of placing sensors all along the pipes of the heat exchanger, it is more common to find sensors at the boundaries of the PHE, i.e. which provide only the inflows and outflows temperatures. Our aim is thus to reconstruct the temperature profiles at each other point in space using only these measured values.

3. OBSERVER DESIGN

Using the fact that controllability and observability of the class of hyperbolic transport systems to which (4) belongs has been proved by (Li, 2010), we can build a boundary observer to estimate the distributed temperatures. The measurements from the sensors at the boundaries of the PHE are used by the observer to correct the initial conditions of the PDE system and thus estimate the distributed values. Additionally, for simplicity, we consider a static boundary input and constant exchange coefficients. The observer proposed in this section is based on a late-lumping strategy (design in the PDE framework), where Lyapunov analysis provides the sufficient conditions for the convergence of the estimation error. The choice of the observer gains and its inclusion into the PDE model are also detailed.

We are dealing with the system of hyperbolic PDEs (4) with boundary conditions (2) and initial conditions (3). The estimated state is denoted as $\hat{T} = [\hat{T}^H \, \hat{T}^C]^T$. Considering that only the boundary temperatures are measured, a natural choice for the observer design is to set the dynamics of the estimator as, $\forall x \in [0, 1]$:

$$\partial_t \hat{T} + A \partial_x \hat{T} = B\hat{T} \tag{5}$$

with boundary conditions, respectively for the hot and cold fluid:

$$\begin{cases} \hat{T}^H(0,t) = T_{in}^H(t) + L_1(T^C(0,t) - \hat{T}^C(0,t)) \\ \hat{T}^C(1,t) = T_{in}^C(t) + L_2(T^H(1,t) - \hat{T}^H(1,t)) \end{cases} \tag{6}$$

The inflows boundary conditions of the observers are thus corrected by the estimation errors of the outflows of the other fluid, weighted by the observer gains L_1 and L_2. The dynamics of the estimation error $\epsilon(x,t) = T(x,t) - \hat{T}(x,t)$ is thus, $\forall x \in [0, 1]$:

$$\partial_t \epsilon + A \partial_x \epsilon = B\epsilon \tag{7}$$

with boundary conditions:

$$\epsilon^H(0,t) = -L_1 \epsilon^C(0,t), \quad \epsilon^C(1,t) = -L_2 \epsilon^H(1,t) \tag{8}$$

The stability conditions for this choice of observer architecture are provided by the following theorem.

Theorem 3.1. Consider the system (4) with boundary conditions (2) and (unknown) initial conditions (3), and where the state matrix A is diagonal and has one positive and one negative eigenvalue. Suppose that there exists a diagonal positive definite matrix $P(x) \in \mathbb{R}^{2\times 2}$ whose entries are functions of x, a diagonal positive definite matrix $M \in \mathbb{R}^{2\times 2}$ and two observer gains L_1 and L_2, such that:

$$L_1^2 \frac{F_1}{E_1} P_1(0) - \frac{F_2}{E_2} P_2(0) \leq 0 \tag{9a}$$
$$L_2^2 \frac{F_2}{E_2} P_2(1) - \frac{F_1}{E_1} P_1(1) \leq 0 \tag{9b}$$
$$-M|A|P(x) + B^T P(x) + P(x) B < -\gamma P(x) \tag{9c}$$

where $P_1(x)$ and $P_2(x)$ are the diagonal components of $P(x)$ and $\gamma > 0$ is a tuning coefficient. Then, the estimator (5) with boundary conditions (6) is a boundary observer for (4).

Proof. We select the Lyapunov function candidate proposed by Diagne et al. (2012):

$$V(\epsilon) = \int_0^1 \epsilon^T P(x) \epsilon \, \mathrm{d}x \tag{10}$$

where $P(x) = \mathrm{diag}\{p_i e^{-\nu_i \mu_i x}\}$ for $i = 1, 2$ such that $p_i > 0$, $\mu_i \in \mathbb{R}^+$ and $\nu_i = sign(A_{ii})$, where A_{ii} is the entry in the i^{th} row and i^{th} column of A. Note that this Lyapunov function candidate is very similar to the one proposed by Xu and Sallet (2002), which in our case would imply to choose $\nu_i = A_{ii}$ and $p_i = 1$.

The time derivative of V along the solutions of (7) with boundary conditions (8) is:

$$\dot{V}(\epsilon) = \int_0^1 \left[\partial_t \epsilon^T P(x)\epsilon + \epsilon^T P(x)\partial_t \epsilon\right] \mathrm{d}x$$
$$= -\int_0^1 \left[\partial_x \epsilon^T A^T P(x)\epsilon + \epsilon^T P(x) A \partial_x \epsilon\right] \mathrm{d}x$$
$$+ \int_0^1 \epsilon^T \left(B^T P(x) + P(x)B\right)\epsilon \,\mathrm{d}x$$

Note that:

$$A^T P(x) = P(x)A^T = \begin{pmatrix} p_1 \frac{F_1}{E_1} e^{-\nu_1 \mu_1 x} & 0 \\ 0 & -p_2 \frac{F_2}{E_2} e^{-\nu_2 \mu_2 x} \end{pmatrix}$$

and:

$$\partial_x \epsilon^T A^T P(x)\epsilon + \epsilon^T P(x) A \partial_x \epsilon$$
$$= \partial_x(\epsilon^T A P(x)\epsilon) + \epsilon^T M|A|P(x)\epsilon$$

with:

$$|A|P(x) = \begin{pmatrix} p_1 \left|\frac{F_1}{E_1}\right| e^{-\nu_1 \mu_1 x} & 0 \\ 0 & p_2 \left|\frac{F_2}{E_2}\right| e^{-\nu_2 \mu_2 x} \end{pmatrix}$$

a positive-definite symmetric matrix and $M = diag\{\mu_1, \mu_2\}$. Then (using integration by parts):

$$\dot{V}(\epsilon) = -\int_0^1 \left[\partial_x(\epsilon^T A P(x)\epsilon) + \epsilon^T M|A|P(x)\epsilon\right] \mathrm{d}x$$
$$+ \int_0^1 \epsilon^T \left(B^T P(x) + P(x)B\right)\epsilon \,\mathrm{d}x$$
$$= -\left[\epsilon^T A P(x)\epsilon\right]_0^1$$
$$+ \int_0^1 \epsilon^T \left(-M|A|P(x) + B^T P(x) + P(x)B\right)\epsilon \,\mathrm{d}x$$

Evaluating the integral at the end-points of the space interval, and replacing the values of the error at the boundaries by their counterparts in equation (8), we get:

$$\dot{V}(\epsilon) = \epsilon^H(1,t)^2 \left[L_2^2 \frac{F_2}{E_2} P_2(1) - \frac{F_1}{E_1} P_1(1)\right] \tag{11}$$
$$+ \epsilon^C(0,t)^2 \left[L_1^2 \frac{F_1}{E_1} P_1(0) - \frac{F_2}{E_2} P_2(0)\right]$$
$$+ \int_0^1 \epsilon^T \left(-M|A|P(x) + B^T P(x) + P(x)B\right)\epsilon \,\mathrm{d}x$$

To ensure the convergence of the error between the actual and the observed states toward zero, $\dot{V}$ needs to be negative. Therefore, the conditions for the convergence of our observer are given by the inequalities (9a)-(9c). ♢

Considering the Lyapunov function (10) with: $P(x) = \mathrm{diag}\{p_i e^{-\nu_i \mu_i x}\}$ for $i = 1, 2$ and using the observer gains to cancel the impact of the boundary conditions on the derivative of the Lyapunov function (11), L_1 and L_2 can be chosen such that (maximum admissible gains according to (9a)-(9b)):

$$L_1^2 \frac{F_1}{E_1} P_1(0) - \frac{F_2}{E_2} P_2(0) = 0 \Leftrightarrow L_1 = \sqrt{\frac{E_1}{F_1}\frac{F_2}{E_2}\frac{P_2(0)}{P_1(0)}} \tag{12a}$$
$$L_2^2 \frac{F_2}{E_2} P_2(1) - \frac{F_1}{E_1} P_1(1) = 0 \Leftrightarrow L_2 = \sqrt{\frac{E_2}{F_2}\frac{F_1}{E_1}\frac{P_1(1)}{P_2(1)}} \tag{12b}$$

The exponential convergence of the integrated squared observation error is then obtained as:

$$\int_0^1 \epsilon(x,t)^T \epsilon(x,t)\,\mathrm{d}x \le \alpha e^{-\gamma t} \int_0^1 \epsilon(x,0)^T \epsilon(x,0)\,\mathrm{d}x$$

with $\alpha = \max_x \{\lambda_{max}(P(x))/\lambda_{min}(P(x))\}$ where λ_{min} and λ_{max} denote the minimum and maximum eigenvalues of the matrix considered, respectively.

Remark 3.1. The design method proposed by (Diagne et al., 2012) consists in finding $P(x)$ such that:

$$B^T P(x) + P(x)B < 0,$$

motivated by the fact that $M|A|P(x)$ can only improve the convergence. Such approach is not applicable here since our B matrix implies that the previous inequality cannot be verified for our candidate $P(x) > 0$. This is easily proved by computing the determinants of $B^T P(x) + P(x)B$.

4. SIMULATION RESULTS

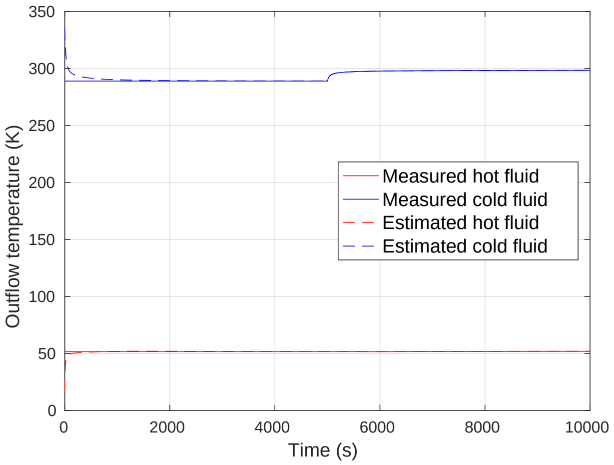
The effectiveness of the proposed observer scheme is illustrated by comparing the outputs of the system's model along with the results of the observer simulation. The PHE model is the one described by equation (1). The corresponding reference simulator is the CEA model provided by Bonne et al. (2014); Bonne (2014), which has been validated with experimental measurements. The PDEs are discretized using Euler schemes, upwind or downwind depending on the flow directions. The transport parameters of the PDE for the observer design are supposed to be constant along the space and time intervals: even though the parameters (mostly h) do depend on the local value of temperature, their change is unnoticeable in the range of temperatures considered. We thus assume the system of PDEs to be linear. The outputs of the system are the temperatures at the boundaries of the heat exchanger, where the sensors are placed in the inflows and outflows.

The observer gains are computed as follows:

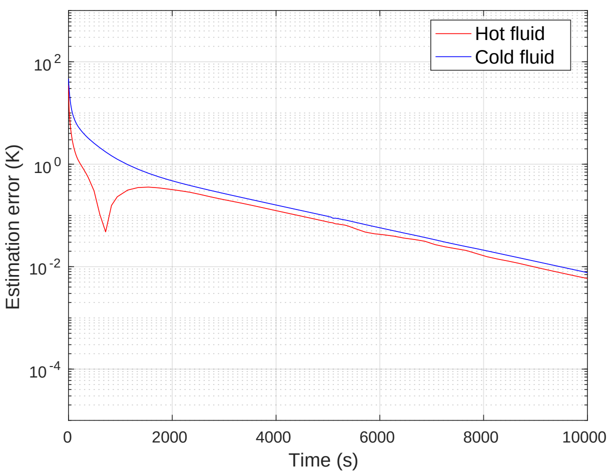
1. the coefficients μ_i are chosen as $|A_{ii}|$;
2. the YALMIP toolbox proposed by Lofberg (2004) provides, if possible, a diagonal matrix $\bar{P} > 0$ satisfying (9c) for the largest value of γ;
3. the coefficients of P satisfying Theorem 3.1 are calculated as $p_i = \bar{P}_{ii} e^{\nu_i \mu_i}$ ($\bar{P}$ thus corresponds to the solution for $x = 1$);
4. we check that (9c) is verified for several values $P(x)$ with a discretized space for $x \in [0\,1]$;
5. L_1 and L_2 are obtained according to (12a)-(12b).

Note that the computation steps 3 to 4 could be replaced by using multiple linear matrix inequalities (LMI) instead of (9c), as suggested by the polytopic approach proposed by Lamare et al. (2016).

The simulations are run with the most extreme initial and inflow temperatures used by CEA: $T^H(x,0) = T^H_{in}(t) = 300\,\mathrm{K}$ and $T^C(x,0) = T^C_{in}(t) = 4\,\mathrm{K}$. $T^H_{in}(t)$ is increased

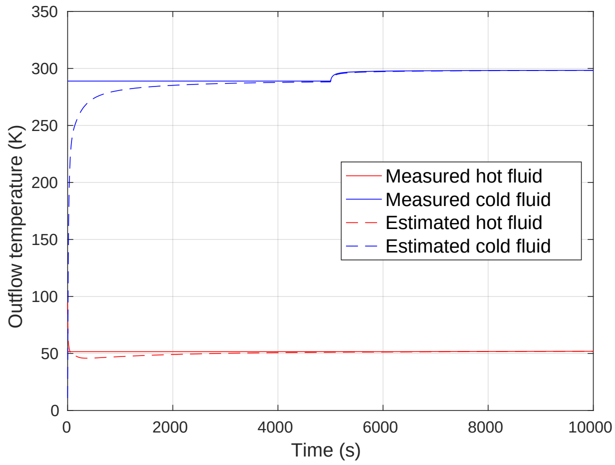


(a) Reference and estimated outflows temperatures.

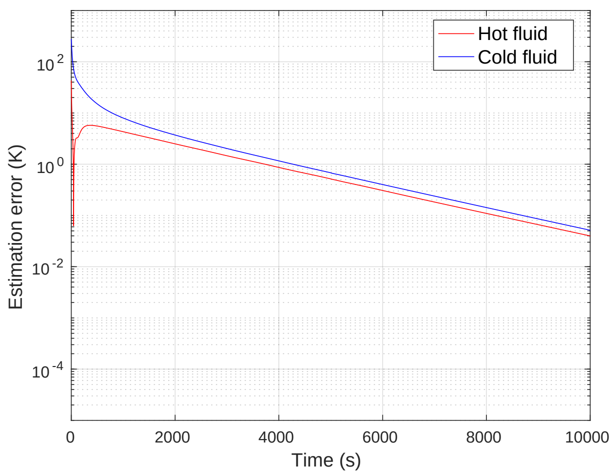


(b) Observation errors.

Fig. 2. Boundary PDE observer for $\hat{T}^H(x,0) = 350\,\mathrm{K}$ and $\hat{T}^C(x,0) = 1\,\mathrm{K}$.



(a) Reference and estimated outflows temperatures.



(b) Observation errors.

Fig. 3. Boundary PDE observer for $\hat{T}^H(x,0) = 10\,\mathrm{K}$ and $\hat{T}^C(x,0) = 100\,\mathrm{K}$.

by 10 K at 5000 s. According to the PHE specifications, the state-space matrices are set with the coefficients $E_1 = 1.82 \times 10^5$, $E_2 = 1.75 \times 10^4$, $F_1 = 311.51$, $F_2 = 312.44$ and $h = 6920$. We obtain the following matrix $P(x)$:

$$P(x) = \begin{bmatrix} 18.77\,e^{-0.0017x} & 0 \\ 0 & 1.75\,e^{0.0178x} \end{bmatrix}$$

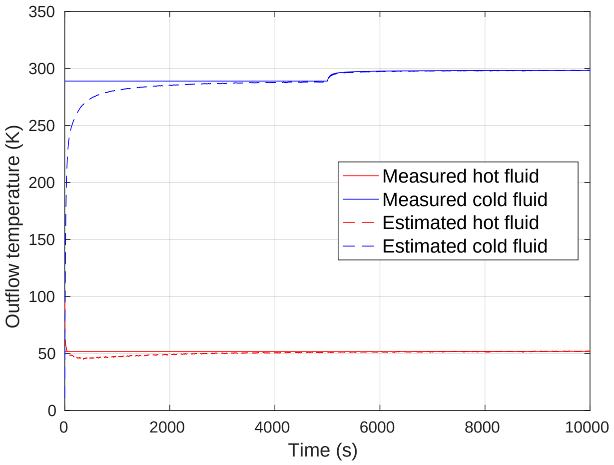
The corresponding observer gains are:

$$L_1 = 0.986, \quad L_2 = 1.004$$

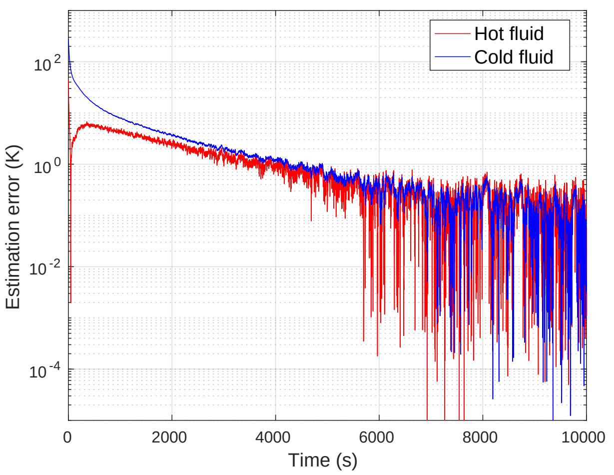
We first initialize the observer at values close to the system's initial conditions $\hat{T}^H(x,0) = 350\,\mathrm{K}$ and $\hat{T}^C(x,0) = 1\,\mathrm{K}$, which provides the plots shown on Fig. 2(a). Note that, as we consider the outflow temperatures the hot channel temperature became cold and the cold channel temperature became hot (the initialization transients are not observable on the figure time-scale). We can see that the observer converges smoothly without initial oscillations. The plots of the errors are shown on Fig. 2(b). It can be seen from those plots that the errors follow an exponential decrease. The error peak toward zero on the hot fluid corresponds to the crossing of the estimated and true values (the estimated value being initially hotter and converging to a colder value).

For the second test, we change the initial conditions and use values very far from the system's initial conditions. For this purpose, we select the observer's initial conditions as $\hat{T}^H(x,0) = 10\,\mathrm{K}$ and $\hat{T}^C(x,0) = 100\,\mathrm{K}$, while maintaining the system's initial conditions at $T^H(x,0) = 300\,\mathrm{K}$ and $T^C(x,0) = 4\,\mathrm{K}$. The plots of the simulation of such system are shown on Fig. 3(a), while the estimation errors are shown on Fig. 3(b). It is obvious that the observer still converges toward the actual value, given enough time. However, with such distant initial conditions, the time to reach the desired values is increased due to an higher initial estimation error (the convergence rate remains the same).

A third scenario investigates the robustness of the method to sensor noise. This is achieved by adding a band-limited white noise (of amplitude 1 K and sampling time 1 s) on the four measurements used by the observer. The simulated



(a) Reference and estimated outflows temperatures.



(b) Observation errors.

Fig. 4. Boundary PDE observer for $\hat{T}^H(x,0) = 10\,\mathrm{K}$ and $\hat{T}^C(x,0) = 100\,\mathrm{K}$ with a band-limited white noise on the measurements.

response is shown on Fig. 4(a)-4(b). We clearly observe the convergence of the estimation error to a minimum value determined by the noise amplitude. The measurement noise is attenuated and we thus conclude that the proposed observer is reasonably robust.

The step on the hot fluid inflow temperature at 5000 s only reflects on the cold outflow temperature and does not influence the observer convergence.

Remark 4.1. This observer has been compared with a classical observer designed using the early lumping approach. While such observer is perfectly able to track the steady-state values of the temperature, its disadvantage is to cause the loss of some of the system's dynamics before the design of the observer (due to the discrete space approximation). Additionally, as we increase the resolution of the space discretization, the rank of the observability matrix decreases due to numerical errors. This inability to find a high resolution solution has led us to adopt the proposed PDE approach.

5. CONCLUSION

The objective of this work is to find an effective way of estimating the temperature, at every point in space and at each instant of time, along the channels of a counter-flow PHE.

Our PDE approach gives satisfactory results in terms of tracking the actual temperatures and of allowing us to freely change the space resolution. It has the advantage of using a Lyapunov method, an efficient technique to ensure the exponential decay of the error function between the actual and the desired state. We used Lyapunov analysis to derive the LMI conditions that ensure the convergence of the observer and have proposed a design method that ensures the exponential convergence of the estimation error. Simulation results illustrate the performance and robustness of the method on a realistic test case. Our result generally applies to the observation of systems with coupled hyperbolic equations describing reverse flows.

An interesting future work would be to extend this approach to quasilinear and time-varying hyperbolic PDEs. It would also be interesting to decouple the two equations and apply the method of characteristics for analytical solutions.

Acknowledgments.

The authors wish to thank Christophe Prieur for the fruitful and interesting discussions concerning the stability analysis for the observer.

REFERENCES

Alinhac, S. (2009). *Hyperbolic Partial Differential Equations*. Universitext. Springer New York.

Bonne, F., Alamir, M., and Bonnay, P. (2014). Nonlinear observer of the thermal loads applied to the helium bath of a cryogenic joule-thompson cycle. *Journal of Process Control*, 24(3), 73–80. URL `http://www.sciencedirect.com/science/article/pii/S0959152413002709`.

Bonne, F. (2014). *Modelling and control of large cryogenic refrigerator*. Ph.d. thesis, Université Grenoble Alpes. URL `https://tel.archives-ouvertes.fr/tel-01133432`.

Castillo, F., Witrant, E., Prieur, C., and Dugard, L. (2012). Dynamic boundary stabilization of linear and quasi-linear hyperbolic systems. In *51st IEEE Conference on Decision and Control (CDC)*, 2952–2957. URL `http://dx.doi.org/10.1109/CDC.2012.6426802`.

Castillo, F., Witrant, E., Prieur, C., and Dugard, L. (2013). Boundary observers for linear and quasi-linear hyperbolic systems with application to flow control. *Automatica*, 49(11), 3180–3188. URL `http://www.sciencedirect.com/science/article/pii/S0005109813003750`.

Coron, J.M., d'Andrea Novel, B., and Bastin, G. (2007). A strict lyapunov function for boundary control of hyperbolic systems of conservation laws. *IEEE Transactions on Automatic Control*, 52(1), 2–11. URL `http://dx.doi.org/10.1109/TAC.2006.887903`.

Di Meglio, F., Vazquez, R., and Krstic, M. (2013). Stabilization of a system of n+1 coupled first-order hyperbolic linear pdes with a single boundary input. *IEEE Transactions on Automatic Control*, 58(12), 3097–3111. URL http://dx.doi.org/10.1109/TAC.2013.2274723.

Diagne, A., Bastin, G., and Coron, J.M. (2012). Lyapunov exponential stability of 1-d linear hyperbolic systems of balance laws. *Automatica*, 48(1), 109–114. URL http://www.sciencedirect.com/science/article/pii/S0005109811004705.

Fratczak, M., Nowak, P., and Czeczot, J. (2014). Simplified modeling of plate heat exchangers. In *Methods and Models in Automation and Robotics (MMAR), 2014 19th International Conference On*, 578–583. URL http://dx.doi.org/10.1109/MMAR.2014.6957418.

Krstic, M. and Smyshlyaev, A. (2008). Backstepping boundary control for first-order hyperbolic {PDEs} and application to systems with actuator and sensor delays. *Systems & Control Letters*, 57(9), 750–758. URL http://www.sciencedirect.com/science/article/pii/S0167691108000352.

Lamare, P.O., Girard, A., and Prieur, C. (2016). An optimisation approach for stability analysis and controller synthesis of linear hyperbolic systems. *ESAIM: COCV*, 22(4), 1236–1263. URL http://dx.doi.org/10.1051/cocv/2016038.

Li, T. (2010). *Controllability and Observability for Quasilinear Hyperbolic Systems*, volume 3. American Institute for Mathematic Sciences, Springfield, Mo.

Lofberg, J. (2004). Yalmip : a toolbox for modeling and optimization in matlab. In *Computer Aided Control Systems Design, 2004 IEEE International Symposium on*, 284–289. URL http://dx.doi.org/10.1109/CACSD.2004.1393890.

Rauch, J. and Taylor, M. (1975). Exponential decay of solutions to hyperbolic equations in bounded domains. *Indiana Univ. Math. J.*, 24, 79–86. URL http://www.iumj.indiana.edu/IUMJ/FULLTEXT/1975/24/24004.

Vazquez, R., Krstic, M., and Coron, J.M. (2011). Backstepping boundary stabilization and state estimation of a 2×2 linear hyperbolic system. In *2011 50th IEEE Conference on Decision and Control and European Control Conference*, 4937–4942. URL https://doi-org.gaelnomade-1.grenet.fr/10.1109/CDC.2011.6160338.

Xu, C.Z. and Sallet, G. (2002). Exponential stability and transfer functions of processes governed by symmetric hyperbolic systems. *ESAIM: COCV*, 7, 421–442. URL http://dx.doi.org/10.1051/cocv:2002062.