Adaptive Stabilization of the Kuramoto-Sivashinsky Equation Subject to Intermittent Sensing

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***Abstract*— We study in this paper the one-dimensional Kuramoto-Sivashinsky equation (KS), subject to intermittent sensing. Namely, we measure the state on a sub-interval of the spatial domain during certain intervals of time, and we measure the state on the remaining sub-interval of space during the remaining intervals of time. As a result, we assign an active control at the boundaries of the spatial domain, and we set a zero boundary condition at the junction of the two spatial sub-intervals. Under the assumption that the destabilizing coefficient is unknown, we design adaptive boundary controllers that guarantee global exponential stability (GES) of the trivial solution in the L^2 norm. Numerical simulations are performed to illustrate our results.**

I. Introduction

In the late 1970, Y. Kuramoto and G. Sivashinsky introduced, independently, the scalar nonlinear partial differential equation (PDE) given by [1], [2]

$$\Sigma \ : \ \partial_t u + u\partial_x u + \lambda_1 \partial_x^2 u + \partial_x^4 u = 0 \quad x \in [0,1], \qquad (1)$$

where $\lambda_1 \in \mathbb{R}$ is known as the destabilizing coefficient.

The KS equation Σ is used to model phase turbulence in reaction-diffusion systems [1] and thermo-diffusive instabilities in laminar flame fronts [2]. It is also used nowadays to model the fluctuations of fluid films on inclined supports [3], [4], plasma instabilities [5], and surface erosion [6].

Boundary stabilization of the trivial solution $\{u = 0\}$ to Σ, in a suitable norm, has attracted some attention within the control community since [7], where boundary feedback controllers are designed for specific values of λ_1. In particular, when λ_1 is unknown, adaptive boundary feedback laws are proposed in [8]. In [9], an integral transformation is proposed to achieve exponential stabilization with arbitrary specified decay rate, provided that the initial condition is sufficiently small, and λ_1 avoids a set of critical values. In [10], under the assumption that λ_1 is smaller than 1, boundary feedback laws are designed in the presence of external perturbations. In [11], boundary controllers are designed to achieve local output feedback stabilization, the output being the right Neumann trace $\partial_x u(t, 1)$, local stabilization is achieved despite the value of λ_1. The aforementioned boundary controllers, either assume λ_1 to be sufficiently small or the initial conditions to be sufficiently close to the origin, in which cases, only boundary measurements are required for the control design. This being said, the boundary stabilization problem, regardless of the value of λ_1 and the range of the initial condition, has not been addressed in existing literature, when the knowledge of u on the entire spatial domain $[0, 1]$ is not available.

In earlier physics literature, however, the problem is studied under some sophisticated and realistic sensing scenarios. For example, in [12], multiple sensors situated at periodically separated spatial points are used. Each sensor measures an average of the state u over a given spatial interval. Furthermore, a boundary controller is designed at the location of each sensor. Another sensing scenario, applied to the Gray-Scott equation, is presented in [13], where the equation is controlled via time-periodic resets of the state, at periodically separated spatial points. Those results are validated via simulations only. Inspired by the aforementioned physics literature, in intermittent sensing scenarios, we identify state variables (or their derivatives) that are measured only on specific spatial sub-domains during specific intervals of time. For example, the KS equation considered in [14], such that, for some $Y \in (0, 1)$, the state u is measured over the spatial domain $[0, Y]$ only during certain time intervals, and that u over the spatial domain $[Y, 1]$ is available only during the remaining time intervals. As a result, boundary controls are imposed at $x = 0$, $x = Y$, and $x = 1$ to globally stabilize the trivial solution, under the assumption that λ_1 is known, but without constraining its value.

In this paper, we generalize the approach proposed in [14] to the case where λ_1 is unknown. In particular, under the sensing scenario proposed in the aforementioned reference, we design active boundary controllers at $x = 0$ and $x = 1$, while maintaining a zero boundary condition at $x = Y$. The proposed feedback law "adaptively" compensates the effect of the unknown parameter λ_1, which can be any bounded function of time. As a result, we are able to guarantee GES of the origin $\{u = 0\}$ in the L^2 norm.

The rest of the paper is organised as follows. The problem formulation is in Section II. Some preliminaries and key intermediate results are in Section III. The main result is in Section IV. Finally, numerical simulations are provided in Section V.

Notation. For $x \in \mathbb{R}^n$, we define $|x| := \sqrt{xx^\top}$, where $x^\top$ is the transpose of x. Depending on the context, a.e. means either almost every or almost everywhere. Let X be a Banach space with a norm $||.||_X$. Let $p \in [1, \infty]$, we denote by $L^p([a, b]; X)$, where $a, b \in \mathbb{R}$, the space of measurable functions $u : [a, b] \to X$, with finite p norm $||.||_p$, where $||u||_p := \left(\int_a^b ||u(t)||_X^p dt\right)^{\frac{1}{p}}$ if $p < \infty$, and

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$||u||_\infty := \text{ess sup}_{t\in[a,b]} ||u(t)||_X$. If $X = \mathbb{R}$, then we write $L^p(a,b)$ instead of $L^p([a,b];\mathbb{R})$. For $k \in \mathbb{N}$, we denote by $H^k(a,b)$ the Sobolev space of functions $f \in L^2(a,b)$, with weak derivatives, up to order k, in $L^2(a,b)$. For $k \in \mathbb{N}\cup\{\infty\}$, we denote by $\mathcal{C}^k(a,b)$ the space of k-times continuously differentiable functions on (a,b). Depending on the context, $g(y)$ denotes either the function g evaluated at a given point y of its domain, or the function g itself. The partial derivative of $f(t,x)$ with respect to t is denoted by $\partial_t f$. The k^{th} partial derivative of $f(t,x)$ with respect to x is denoted by $\partial_x^k f$. We denote the time derivative of a function V either by $\frac{d}{dt}V$ or $\dot{V}$. We may denote the derivative of a function of a scalar variable g by g', and its second derivative by g''. For a function of two variables $f(t,x)$, $f(x)$ denotes the function $t \mapsto f(t,x)$. For $x \in \mathbb{R}$, $\text{sgn}(x) = 1$ if $x > 0$, $= 0$ if $x = 0$ and $= -1$ if $x < 0$.

II. Problem Formulation

Consider the KS equation Σ under the following sensing scenario.

A. Intermittent Sensing

Following [14], we let $Y \in (0,1)$ and we assume that $u([0,Y])$ is measured during certain intervals of time, and that $u([Y,1])$ is measured during the remaining ones. More precisely, we assume that there exists a sequence of time instants $\{t_i\}_{i=1}^\infty$, with $t_1 = 0$ and $t_{i+1} > t_i$, such that

- $u([0,Y])$ is available a.e. in $I_1 := \bigcup_{k=1}^\infty [t_{2k-1}, t_{2k})$.
- $u([Y,1])$ is available a.e. in $I_2 := \bigcup_{k=1}^\infty [t_{2k}, t_{2k+1})$.

Associated to this intermittent sensing, we consider the following dwell-time condition.

Condition 1: There exist four constants $\overline{T}_1, \overline{T}_2, \underline{T}_1, \underline{T}_2 > 0$ such that, for each $k \in \{1,2,...\}$, we have

$$\underline{T}_1 \leq t_{2k} - t_{2k-1} \leq \overline{T}_1 \quad \text{and} \quad \underline{T}_2 \leq t_{2k+1} - t_{2k} \leq \overline{T}_2.$$

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B. Boundary Control Locations

We propose to control Σ at three different locations: at $x = 0$, $x = Y$, and $x = 1$. We, therefore, assimilate Σ to a system of two KS equations, interconnected by a boundary constraint at $x = Y$, given by

$$\Sigma_2 : \begin{cases} \partial_t w + w\partial_x w + \lambda_1 \partial_x^2 w + \partial_x^4 w = 0, & x \in [0,Y], \\ \partial_t v + v\partial_x v + \lambda_1 \partial_x^2 v + \partial_x^4 v = 0, & x \in [Y,1]. \end{cases}$$

The boundary conditions imposed, for almost all $t \geq 0$, are

$$\begin{aligned} & w(Y) = v(Y) = \partial_x w(Y) = \partial_x v(Y) = 0, \\ & \partial_x w(0) = \partial_x v(1) = 0, \\ & w(0) = u_1, \quad v(1) = u_2. \end{aligned} \tag{2}$$

where u_1 and u_2 are control inputs to be designed.

Remark 1: The boundary conditions $w(Y) = v(Y)$ and $\partial_x w(Y) = \partial_x v(Y)$ mean that, for almost all $t \geq 0$, any function, whose restriction to $[0,Y]$ is w and whose restriction to $[Y,1]$ is v, is continuously differentiable at $x = Y$.

Remark 2: Under the boundary conditions in (2), we say that the control at $x = Y$ is passive (since the boundary conditions at this location are set to zero), and that the control actions at $x = 0$ and at $x = 1$ are active.

Before stating our control goals, we first specify the concept of solutions to Σ by, first, specifying the solutions to Σ_2.

Definition 1 (Solution to Σ_2): Given an initial condition $(w_o, v_o) \in H^4(0,Y) \times H^4(Y,1)$, a corresponding solution to Σ_2 is any pair $(w,v) \in L^2([T,T+1];H^4(0,Y)) \times L^2([T,T+1];H^4(Y,1))$ for all $T \geq 0$ such that:

1) $w(t=0) = w_o$ and $v(t=0) = v_o$ a.e. in space;
2) the boundary conditions (2) are satisfied a.e. in time;
3) the pair (w,v) admit weak time derivatives;
4) the equations of Σ_2 are satisfied a.e. in space and time.

Now, we specify the concept of solutions to Σ.

Definition 2 (Solution to Σ): A function $u : \mathbb{R}_{\geq 0} \times [0,1] \to \mathbb{R}$ is said to be a solution to Σ if there exists a solution (w,v) to Σ_2 such that for a.e. $t \in \mathbb{R}_{\geq 0}$, $u(t,x) = w(t,x)$ a.e. in $[0,Y]$ and $u(t,x) = v(t,x)$ a.e. in $[Y,1]$.

Remark 3: According to Definition 1, the availability of $w(t,[0,Y])$ for a.e. $t \in I_1$ implicitly suggests the availability of $w_x^k(t,[0,Y])$, $k \in \{1,2,3\}$, for a.a. $t \in I_1$. Similarly, the availability of $v(t,[Y,1])$ for a.e. $t \in I_2$ implicitly suggests the availability of $v_x^k(t,[Y,1])$, $k \in \{1,2,3\}$, for a.e. $t \in I_2$.

C. Control Objective

Our goal is to globally stabilize, in the L^2 norm, the trivial solution to Σ, under the proposed intermittent sensing scenario, while considering the situation described in the following assumption.

Assumption 1: The coefficient λ_1 is strictly positive, and both λ_1 and the constants $(\overline{T}_1, \overline{T}_2, \underline{T}_1, \underline{T}_2)$ in Condition 1 are *unknown*. •

To address the latter two problems, when $t \in I_1$, we design (u_1, u_2) to stabilize the dynamics of w (defined on $[0,Y]$), while maintaining an appropriate behavior for v (which evolves on $[Y,1]$). The same reasoning applies when $t \in I_2$, *mutatis mutandis*.

III. Preliminaries

In this section, we introduce preliminary results that play a key role to prove our main results.

To start, we use Lions-Magenes Lemma (see for e.g. [15], Page 106, Proposition 1.2) to conclude that, due to the used space of solutions and the structure of Σ, the maps $t \mapsto \int_0^Y w(x)^2 dx$ and $t \mapsto \int_Y^1 v(x)^2 dx$ are locally absolutely continuous and that the Leibniz integral rule holds a.e. in time. This means that, for a.e. $t \in \mathbb{R}_{\geq 0}$, we have

$$\frac{d}{dt}\int_0^Y w(x)^2 dx = 2\int_0^Y \partial_t w(x)\ w(x)\ dx,$$
$$\frac{d}{dt}\int_Y^1 v(x)^2 dx = 2\int_Y^1 \partial_t v(x)\ v(x)\ dx.$$

Next, we recall a key inequality that links the L^2 norm of a function with the L^2 norms of its first and second derivatives. This inequality has been introduced under different forms in [16], [17], [18], [19]. The finest version is recalled in the following lemma.

Lemma 1 ([20], page 84, inequality 23.1): Let $b > a > 0$ and $f \in \mathcal{C}^2(a,b)$. Then, for each $\epsilon > 0$, we have

$$\int_a^b f'(x)^2 dx \leq \left[\frac{P}{\epsilon} + \frac{Q}{(b-a)^2}\right]\int_a^b f(x)^2 dx + \epsilon\int_a^b f''(x)^2 dx. \tag{3}$$

where $P := 1$ and $Q := 12$. Moreover, if $P < 1$ or $Q < 12$, then (3) cannot hold for all $f \in \mathcal{C}^2(a,b)$ and for all $\epsilon > 0$. □

Using Lemma 1, we are able to prove the following result.

Lemma 2: Along each pair (w, v) solution to Σ_2 the Lyapunov function candidates

$$V_1(w) := \frac{1}{2}\int_0^Y w(x)^2 dx \text{ and } V_2(v) := \frac{1}{2}\int_Y^1 v(x)^2 dx$$

verify, for a.a. $t \geq 0$,

$$\begin{aligned}\dot{V}_1 \leq\ & \theta_1 V_1 - \frac{w(Y)^3 - w(0)^3}{3} - \lambda_1 w(Y)\partial_x w(Y)\\ &+ \lambda_1 w(0)\partial_x w(0) - w(Y)\partial_x^3 w(Y) + w(0)\partial_x^3 w(0)\\ &+ \partial_x w(Y)\partial_x^2 w(Y) - \partial_x w(0)\partial_x^2 w(0),\end{aligned}$$

$$\begin{aligned}\dot{V}_2 \leq\ & \theta_2 V_2 - \frac{v(1)^3 - v(Y)^3}{3}\lambda_1 v(1)\partial_x v(1) + \lambda_1 v(Y)\partial_x v(Y)\\ &- v(1)\partial_x^3 v(1) + v(Y)\partial_x^3 v(Y)\\ &+ \partial_x v(1)\partial_x^2 v(1) - \partial_x v(Y)\partial_x^2 v(Y),\end{aligned}$$

where $\theta_1 := 2\lambda_1\left(\lambda_1 + \frac{12}{Y^2}\right)$ and $\theta_2 := 2\lambda_1\left(\lambda_1 + \frac{12}{(1-Y)^2}\right)$. □

Using Lemma 2 and the boundary conditions in (2), we obtain the following differential inequalities for (V_1, V_2) along the solutions to Σ_2, which hold for a.e. $t \geq 0$.

$$\begin{aligned}\dot{V}_1 &\leq \theta_1 V_1 + \frac{u_1^3}{3} + u_1\partial_x^3 w(0),\\ \dot{V}_2 &\leq \theta_2 V_2 - \frac{u_2^3}{3} - u_2\partial_x^3 v(1).\end{aligned} \tag{4}$$

IV. Main result

For system Σ_2, the differential inequalities in (4) become

$$\begin{aligned}\dot{V}_1 &\leq \theta_1 V_1 + \frac{u_1^3}{3} + u_1\partial_x^3 w(0),\\ \dot{V}_2 &\leq \theta_2 V_2 - \frac{u_2^3}{3} - u_2\partial_x^3 v(1).\end{aligned} \tag{5}$$

Next, we show how to design the control inputs (u_1, u_2).

A. Control Design

Given two functions $\hat{\theta}_1$, $\hat{\theta}_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ to be designed,

- on I_1, we let $u_2 = 0$ and we choose u_1 such that
$$u_1^3 + 3u_1\partial_x^3 w(0) \leq -3\hat{\theta}_1 V_1. \tag{6}$$
- on I_2, we let $u_1 = 0$ and we choose $u_2 = 0$ such that
$$-u_2^3 - 3u_2\partial_x^3 v(1) \leq -3\hat{\theta}_2 V_2. \tag{7}$$

In the following lemma, we propose an explicit design of the control laws u_1 and u_2 to satisfy (6) and (7), respectively.

Lemma 3: To satisfy (6), we set $u_1 := \kappa(V_1, \partial_x^3 w(0), \hat{\theta}_1)$, where

$$\kappa(\cdot) := \begin{cases} -\operatorname{sgn}(\partial_x^3 w(0))V_1^{\frac{1}{3}} & \text{if } |\partial_x^3 w(0)| \geq l(V_1, \hat{\theta}_1),\\ k(V_1, \hat{\theta}_1) & \text{otherwise.}\end{cases}$$

where $l(V_1, \hat{\theta}_1) := (1/3)[1 + 3\hat{\theta}]V_1^{\frac{2}{3}}$ and k is bounded for bounded arguments, and satisfies

$$k(V_1, \hat{\theta}_1)^3 + 3|k(V_1, \hat{\theta}_1)|l(V_1, \hat{\theta}_1) \leq -3\hat{\theta}_1 V_1. \tag{8}$$

To satisfy (7), we set $u_2 := -\kappa(V_2, \partial_x^3 v(1), \hat{\theta}_2)$. □

Proof: We distinguish between the two cases. If $|\partial_x^3 w(0)| \geq l(V_1, \hat{\theta}_1)$, then

$$\frac{u_1^3}{3} + u_1\partial_x^3 w(0) \leq \frac{V_1^3}{3} - V_1 l(V_1, \hat{\theta}_1) \leq -\hat{\theta}_1 V_1.$$

Otherwise, we have

$$\begin{aligned}\frac{u_1^3}{3} + u_1\partial_x^3 w(0) &\leq \frac{k(V_1, \hat{\theta}_1)^3}{3} + |k_1(V_1, \hat{\theta}_1)|l(V_1, \hat{\theta}_1)\\ &\leq -\hat{\theta}_1 V_1.\end{aligned}$$

The same reasoning applies for u_2. ■

Remark 4: To verify (8), we can choose

$$k(V_1, \hat{\theta}_1) := -3[3\hat{\theta}_1 + 1]V_1^{1/3}.$$

Remark 5: The choice of the control laws in Lemma 3 guarantees that $u_1 = 0$ (respectively, $u_2 = 0$) whenever $V_1 = 0$ (respectively, $V_2 = 0$). More importantly, the map κ is bounded in the second argument, even if this one is not guaranteed to remain bounded. As a result, we guarantee boundedness of (u_1, u_2) in closed loop provided that (V_1, V_2) and $(\hat{\theta}_1, \hat{\theta}_2)$ are bounded.

Now, we illustrate how to design the adaptation parameters $\hat{\theta}_1$ and $\hat{\theta}_2$, which compensate the effect of the destabilizing terms $\theta_1 V_1$ and $\theta_2 V_2$ in (5). Roughly speaking, $\hat{\theta}_1$ (respectively $\hat{\theta}_2$) is dynamically defined, for a given $\sigma > 0$, as a strictly-increasing function until we notice an exponential decrease of V_1 (respectively V_2) at the rate $\sigma > 0$. In which case, we freeze the value of $\hat{\theta}_1$ (respectively $\hat{\theta}_2$). Note that $\hat{\theta}_1$ (respectively $\hat{\theta}_2$) will end up being bounded, since after $\hat{\theta}_1$ (respectively $\hat{\theta}_2$) exceeds θ_1 (respectively, θ_2), V_1 (respectively V_2), along solutions, will decrease exponentially at a rate in the range of $\hat{\theta}_1 - \theta_1$ (respectively, $\hat{\theta}_2 - \theta_2$). Strictly speaking, the behavior of $(\hat{\theta}_1, \hat{\theta}_2)$ is governed by the following algorithm.

Task 1. On each interval $[t_{2k-1}, t_{2k}) \subset I_1$, we set $\dot{\hat{\theta}}_2 = 0$ and, if $V_1(t_{2k-1}) > V_1(t_{2k-3})e^{-\sigma(t_{2k-1}-t_{2k-3})}$ we set $\dot{\hat{\theta}}_1 := \Delta_1 > 0$, otherwise we set $\dot{\hat{\theta}}_1 := 0$.

Task 2. On each interval $[t_{2k}, t_{2k+1}) \subset I_2$, we set $\dot{\hat{\theta}}_1 := 0$ and, if $V_2(t_{2k}) > V_2(t_{2k-2})e^{-\sigma(t_{2k}-t_{2k-2})}$ we set $\dot{\hat{\theta}}_2 := \Delta_2 > 0$, otherwise we set $\dot{\hat{\theta}}_2 := 0$.

Task 3. The initial conditions are non negative, i.e. $\hat{\theta}_1(0) \geq 0$ and $\hat{\theta}_2(0) \geq 0$, and, on the interval $[t_1, t_3]$, we set $\hat{\theta}_1 = \hat{\theta}_1(0)$ and $\hat{\theta}_2 = \hat{\theta}_2(0)$.

B. Closed-Loop Analysis

To analyse the resulting closed-loop system, we introduce the following Lemma.

Lemma 4: Consider the switched differential inequalities

$$\begin{cases} \dot{V}_1 \leq & (\theta_1 - \hat{\theta}_1)V_1 \\ \dot{V}_2 \leq & \theta_2 V_2 \end{cases} \quad \text{a.e. in } I_1, \\ \begin{cases} \dot{V}_1 \leq & \theta_1 V_1 \\ \dot{V}_2 \leq & (\theta_2 - \hat{\theta}_2)V_2 \end{cases} \quad \text{a.e. in } I_2. \tag{9}$$

where $(V_1, V_2) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$, θ_1 and θ_2 are positive constants, $I_1 := \cup_{k=1}^{\infty}[t_{2k-1}, t_{2k})$, and $I_2 := \cup_{k=1}^{\infty}[t_{2k}, t_{2k+1})$, with $\{t_i\}_{i=1}^{\infty}$ a sequence of time instants such that $t_1 = 0$ and $t_{i+1} > t_i$, and $\hat{\theta}_1, \hat{\theta}_2$ are defined, for some $\sigma > 0$, according to Task 1.-Task 3.. Furthermore, suppose that there exist positive constants $\overline{T}_1, \overline{T}_2, \underline{T}_1$, and $\underline{T}_2$ such that Condition 1 holds. Then, there exists a positive constant κ such that, for each locally absolutely continuous solution (V_1, V_2) to (9), we have that $(\hat{\theta}_1, \hat{\theta}_2)$ is bounded, and

$$V_1(t) + V_2(t) \leq \kappa(V_1(0) + V_2(0))e^{-\sigma t} \qquad \forall t \geq 0. \tag{10}$$

□

Proof: The proof follows in two steps. First, we prove that $(\hat{\theta}_1, \hat{\theta}_2)$ are bounded by showing that they become constant after some finite time $T > 0$. The second step shows that the Lyapunov function candidate $W := V_1 + V_2$ decays exponentially to zero.

To prove that $(\hat{\theta}_1, \hat{\theta}_2)$ become constants after some finite time $T > 0$, we use contradiction. That is, we assume that there is no finite time $T > 0$ such that $\dot{\hat{\theta}}_1(t) = \dot{\hat{\theta}}_2(t) = 0$ for all $t \geq T$. This means, according to Task 1.-Task 2. that there exists an infinite number of time intervals, each one having a length greater or equal than $\min\{\underline{T}_1, \underline{T}_2\}$, on which, $\hat{\theta}_1$ and $\hat{\theta}_2$ are linearly increasing, and thus $\lim_{t\to\infty} \hat{\theta}_1(t) = \lim_{t\to\infty} \hat{\theta}_2(t) = \infty$.

Let $\tilde{\theta}_i := \theta_i - \hat{\theta}_i$. It follows that there must exists $k' \in \mathbb{N}^*$ such that, for all integers $k \geq k'$, we have $\tilde{\theta}_1(t_{2k-3}) < 0$, $\tilde{\theta}_2(t_{2k-2}) < 0$, and

$$\begin{aligned} \tilde{\theta}_1(t_{2k-3})\underline{T}_1 + \theta_1\overline{T}_1 \leq -\sigma(\overline{T}_1 + \overline{T}_2), \\ \tilde{\theta}_2(t_{2k-2})\underline{T}_2 + \theta_2\overline{T}_2 \leq -\sigma(\overline{T}_1 + \overline{T}_2). \end{aligned} \tag{11}$$

As a consequence of (11), for all $k \geq k'$, we have

$$\begin{aligned} &\tilde{\theta}_1(t_{2k-3})[t_{2k-2} - t_{2k-3}] + \theta_1[t_{2k-1} - t_{2k-2}] \\ &\qquad \leq -\sigma[t_{2k-1} - t_{2k-3}], \\ &\tilde{\theta}_2(t_{2k-2})[t_{2k-1} - t_{2k-2}] + \theta_2[t_{2k} - t_{2k-1}] \\ &\qquad \leq -\sigma[t_{2k} - t_{2k-2}]. \end{aligned} \tag{12}$$

Using Grönwall-Bellman inequality, for all $k \geq 3$, we obtain

$$\begin{aligned} V_1(t_{2k-1}) &\leq V_1(t_{2k-3})e^{\int_{t_{2k-3}}^{t_{2k-2}} \tilde{\theta}_1(t)dt + \theta_1[t_{2k-1}-t_{2k-2}]} \\ &\leq V_1(t_{2k-3})e^{\tilde{\theta}_1(t_{2k-3})[t_{2k-2}-t_{2k-3}] + \theta_1[t_{2k-1}-t_{2k-2}]}, \\ V_2(t_{2k}) &\leq V_2(t_{2k-2})e^{\int_{t_{2k-2}}^{t_{2k-1}} \tilde{\theta}_2(t)dt + \theta_2[t_{2k}-t_{2k-1}]} \\ &\leq V_2(t_{2k-2})e^{\tilde{\theta}_2(t_{2k-2})[t_{2k-1}-t_{2k-2}] + \theta_2[t_{2k}-t_{2k-1}]}. \end{aligned} \tag{13}$$

From (12)-(13), we conclude that, for all $k \geq k'$, we have

$$\begin{aligned} V_1(t_{2k-1}) \leq V_1(t_{2k-3})e^{-\sigma[t_{2k-1}-t_{2k-3}]}, \\ V_2(t_{2k}) \leq V_2(t_{2k-2})e^{-\sigma[t_{2k}-t_{2k-2}]}. \end{aligned}$$

By induction, the latter implies that $\hat{\theta}_1$ is constant for all $t \geq t_{2k'-1}$ and $\hat{\theta}_2$ is constant for all $t \geq t_{2k'}$, which yields to a contradiction.

We analyze now the Lyapunov function candidate W. For this purpose, we define the sequences of time instants $\{T_i\}_{i=0}^{\infty}$ and $\{T'_i\}_{i=1}^{\infty}$, such that $T_i := t_{2i+1}$ and $T'_i := t_{2i}$. In particular, we note that, for all $i \in \{0, 1, 2, ...\}$, we have

$$\begin{aligned} \underline{T}_1 + \underline{T}_2 \leq T_{i+1} - T_i \leq \overline{T}_1 + \overline{T}_2, \\ \underline{T}_1 + \underline{T}_2 \leq T'_{i+2} - T'_{i+1} \leq \overline{T}_1 + \overline{T}_2. \end{aligned}$$

Let $\tau_1 \in \{T_i\}_{i=0}^{\infty}$ be the smallest time instant from which $\hat{\theta}_1$ is constant, and $\tau_2 \in \{T'_i\}_{i=1}^{\infty}$ be the smallest time instant from which $\hat{\theta}_2$ is constant. For all $i \in \{1, 2, ...\}$, such that $T_i \geq \tau_1$ and $T'_i \geq \tau_2$, we have

$$\begin{aligned} V_1(T_{i+1}) \leq V_1(T_i)e^{-\sigma(T_{i+1}-T_i)}, \\ V_2(T'_{i+1}) \leq V_2(T'_i)e^{-\sigma(T'_{i+1}-T'_i)}. \end{aligned}$$

By induction, for all $i \in \{1, 2, ...\}$ such that $T_i \geq \tau_1$ and for all $t \in [T_i, T_{i+1}]$, we have

$$V_1(t) \leq V_1(T_i)e^{\theta_1(\overline{T}_1+\overline{T}_2)} \leq V_1(\tau_1)e^{\theta_1(\overline{T}_1+\overline{T}_2)}e^{-\sigma(T_i-\tau_1)}. \tag{14}$$

Similarly, for all $i \in \{1, 2, ...\}$ such that $T'_i \geq \tau_2$ and for all $t \in [T'_i, T'_{i+1}]$, we have

$$V_2(t) \leq V_2(\tau_2)e^{\theta_2(\overline{T}_1+\overline{T}_2)}e^{-\sigma(T'_i-\tau_2)}. \tag{15}$$

In (14) and (15), we, respectively, use the inequalities

$$\begin{aligned} &e^{-\sigma(T_i-\tau_1)} \leq e^{-\sigma(t-\tau_1)}e^{\sigma(t-T_i)} \leq e^{-\sigma(t-\tau_1)}e^{\sigma(\overline{T}_1+\overline{T}_2)} \\ &e^{-\sigma(T'_i-\tau_2)} \leq e^{-\sigma(t-\tau_2)}e^{\sigma(\overline{T}_1+\overline{T}_2)}. \end{aligned}$$

We can, therefore, rewrite (14) and (15), respectively, as

$$\begin{aligned} V_1(t) \leq V_1(\tau_1)e^{(\theta_1+\sigma)(\overline{T}_1+\overline{T}_2)}e^{-\sigma(t-\tau_1)} \quad \forall t \geq \tau_1, \\ V_2(t) \leq V_2(\tau_2)e^{(\theta_2+\sigma)(\overline{T}_1+\overline{T}_2)}e^{-\sigma(t-\tau_2)} \quad \forall t \geq \tau_2. \end{aligned} \tag{16}$$

Next, by observing that

$$V_1(\tau_1) \le V_1(0)e^{\theta_1\tau_1} \quad \text{and} \quad V_2(\tau_2) \le V_2(0)e^{\theta_2\tau_2},$$

we can re-express (16) as

$$\begin{aligned} V_1(t) &\le V_1(0)e^{(\theta_1+\sigma)(\overline{T}_1+\overline{T}_2+\tau_1)}e^{-\sigma t} \quad \forall t \ge \tau_1, \\ V_2(t) &\le V_2(0)e^{(\theta_2+\sigma)(\overline{T}_1+\overline{T}_2+\tau_2)}e^{-\sigma t} \quad \forall t \ge \tau_2. \end{aligned} \tag{17}$$

Moreover, we have that

$$\begin{aligned} V_1(t) &\le V_1(0)e^{(\theta_1+\sigma)\tau_1}e^{-\sigma t} \quad \forall t \le \tau_1, \\ V_2(t) &\le V_2(0)e^{(\theta_2+\sigma)\tau_2}e^{-\sigma t} \quad \forall t \le \tau_2. \end{aligned} \tag{18}$$

Finally, by denoting,

$$\mu := \max\left(e^{(\theta_1+\sigma)(\overline{T}_1+\overline{T}_2+\tau_1)}; e^{(\theta_2+\sigma)(\overline{T}_1+\overline{T}_2+\tau_2)}\right),$$

and based on (17) and (18), we obtain $W(t) \le \mu W(0)e^{-\sigma t}$ for all $t \ge 0$. To complete the proof, we show that μ can be upper-bounded by a constant independent on $(V_1(0), V_2(0))$. This can be done by proving that the time instants (τ_1, τ_2) are independent on $(V_1(0), V_2(0))$.

Note that, for each $t \in [t_{2k-1}, t_{2k}] \subset I_1$, we have $\dot{V}_1(t) \le \left(\theta_1 - \hat{\theta}_1(t_{2k-1})\right) V_1(t)$, and, for each $[t_{2k}, t_{2k+1}]$, we have $\dot{V}_1(t) \le \theta_1 V_1(t)$. Let $\overline{V}_1$ be the locally absolutely continuous solution to the switched system

$$\begin{cases} \dot{\overline{V}}_1 = \left(\theta_1 - \hat{\theta}_1(t_{2k-1})\right)\overline{V}_1 & \text{for a.e. } t \in I_1, \\ \dot{\overline{V}}_1 = \theta_1 V_1 & \text{for a.e. } t \in I_2, \end{cases}$$

starting from the initial condition $\overline{V}_1(0) = V_1(0)$, and $\hat{\theta}_1$ designed as in Task 1. while using $\overline{V}_1$ instead of V_1, and $\hat{\theta}_1(0) > 0$. Let $\bar{\tau}_1 \in \{t_{2i+1}\}_{i=0}^{\infty}$ be the smallest time instant, from which, $\hat{\theta}_1$ is constant. From previous computations, we know that $\overline{V}_1(t) \le e^{(\theta_1+\sigma)(\overline{T}_1+\overline{T}_2+\bar{\tau}_1)}\overline{V}_1(0)e^{-\sigma t}$ for all $t \ge 0$. The time instant $\bar{\tau}_1$ is independent on $\overline{V}_1(0)$ since the rate of convergence $\overline{V}_1$ depends only $\theta_1 - \hat{\theta}_1(t_{2k+1})$, θ, and the intervals I_1 and I_2. It also means that $\hat{\theta}_1$ can be seen as a function of time only. Let us now note that the dynamical map $(V_1, t) \mapsto \begin{cases} \left(\theta_1 - \hat{\theta}_1(t_{2k-1})\right) V_1 & \text{if } t \in I_1 \\ \theta_1 V_1 & \text{if } t \in I_2 \end{cases}$
is locally Lipshitz, which means that, under the continuity of V_1 and $\overline{V}_1$, $V_1(t) \le \overline{V}_1(t) \le e^{(\theta_1+\sigma)(\overline{T}_1+\overline{T}_2+\bar{\tau}_1)}\overline{V}_1(0)e^{-\sigma t}$ for all $t \ge 0$. The same reasoning applies for V_2, which concludes the proof. ■

We are now ready to state our main result.

Theorem 1: Consider system Σ under the sensing scenario in Section II-A. Let Condition 1 and Assumption 1 hold. Under the boundary conditions in (2), we let $(u_1, u_2) := (\kappa(V_1, \partial_x^3 w(0), \hat{\theta}_1), 0)$ on I_1 and $(u_1, u_2) := (0, -\kappa(V_2, \partial_x^3 v(1), \hat{\theta}_2))$ on I_2, where $\kappa(\cdot)$ is introduced in Lemma 3, and $(\hat{\theta}_1, \hat{\theta}_2)$ are designed, for some $\sigma > 0$, according to Task 1.-Task 3.. Then, the set $\mathcal{A} := \{(u, \hat{\theta}_1, \hat{\theta}_2) : u = 0\}$ is L^2-GES. Namely, for each $\hat{\theta}_1(0)$ and $\hat{\theta}_2(0)$, there exists $\kappa > 0$ such that, for each solution u to Σ with initial condition u_o, we have $||u(t)||_2 \le \kappa ||u_o||_2 e^{-\frac{1}{2}\sigma t}$ for all $t \ge 0$. Moreover, u_1 and u_2 are bounded and converge to zero. □

Proof: When (6) and (7) are satisfied, we conclude that the Lyapunov functions (V_1, V_2), along the solutions to Σ_2, form a solution to (9). Hence, using Lemma 4, we conclude that the pair (V_1, V_2) satisfies (10). As a result, the set $\mathcal{A}$ is L^2-GES. Concerning the boundedness of the control inputs, it is, due to the choice of κ in Lemma 3, a direct consequence of the boundedness of the Lyapunov function candidates V_1 and V_2 and of the adaptation parameters $\hat{\theta}_1$ and $\hat{\theta}_2$. Similarly, the asymptotic convergence of (u_1, u_2) to zero is a straightforward consequence of the convergence of (V_1, V_2) to zero. ■

V. SIMULATIONS

The numerical scheme that we use to simulate Σ in closed loop is an adaptation of the mesh-free collocation method using radial basis functions (RBF)s; see [21]. The first- and the third-order spatial derivatives that appear in the control laws are calculated using Euler forward and Euler backward schemes. The Lyapunov functions V_1 and V_2 are calculated via Riemannian sums. Furthermore, we use multiquaric RBFs, which depend on a shape parameter $c \in \mathbb{R}$. To make sure that the simulation is depicting the actual behavior of Σ in closed-loop, we use the same shape parameter (namely, $c = 0.4$) when simulating both the closed-loop and the open-loop systems. The control input is delayed with a single time step. The simulations are performed on Matlab® R2022b[1].

For the obtained simulations, the initial time is $t_o = 0$, the final time is $t_f = 8 \times 10^{-3}$, and the time step is $\Delta t = 10^{-7}$. We select $N + 1$ uniformly distributed collocation points on the interval $[0, Y]$, with $Y = 0.5$, from $x_o = 0$ to $x_N = Y$, we select the same number of collocation points on $[Y, 1]$, where $N = 9$, which yields to the space-discretisation step $\Delta x \approx 0.0556$. We select the anti-diffusion parameter $\lambda_1 = 4\pi^2/0.25 + 50$, for which, the linearized KS equation is unstable [7]. The initial condition u_o is given by $u_o(x) = -3(\cos(4\pi x) - 1)$ for all $x \in [0, 1]$. The sequences of time intervals I_1 and I_2 are given by : $I_1 = [0, 1) \cup [2, 2.8) \cup [3.9, 5) \cup [5.5, 6.5) \cup [7, 7.6) \times 10^{-3}$ and $I_2 = [1, 2) \cup [2.8, 3.9) \cup [5, 5.5) \cup [6.5, 7) \cup [7.6, 8) \times 10^{-3}$. We set $\hat{\theta}_1(0) = \hat{\theta}_2(0) = 0$ and we choose a linear increment, i.e. on the time intervals where $\hat{\theta}_{1,2}$ should be increasing, we take $\hat{\theta}_{1,2}(n) = \hat{\theta}_{1,2}(n-1) + \Delta$, where $\Delta = 0.01$. Finally, the decay rate required is $\sigma = 100$.

The open-loop response, which is unstable, is shown in Figure 1. The convergence to zero happens in closed-loop as it can be seen in Figure 2. The inputs (u_1, u_2) are discontinuous boundary controllers at $x = 0$ and $x = 1$, respectively; see Figure 3. Moreover, it is interesting to observe that the continuous phase of u_1 (i.e. when $|\partial_x^3 w(0)| < l(V_1, \hat{\theta}_1)$) happens only on $[0, 1]$. The Lyapunov function candidates (V_1, V_2) and their sum $W = V_1 + V_2$, along the closed-loop solutions, are illustrated in Figure 4. In this last figure, we can see the impact of the intermittent sensing as V_1 (resp. V_2)

[1]The simulation code can be found at https://github.com/BelhadjoudjaMohamedCamil/Kuramoto-Sivashinsky23.git

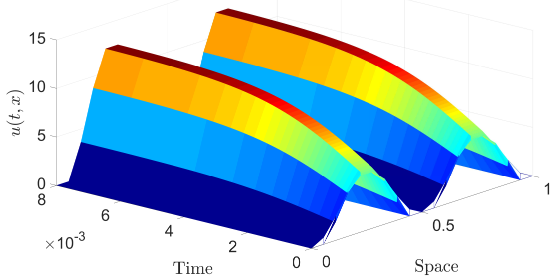

Fig. 1: Open-loop response.

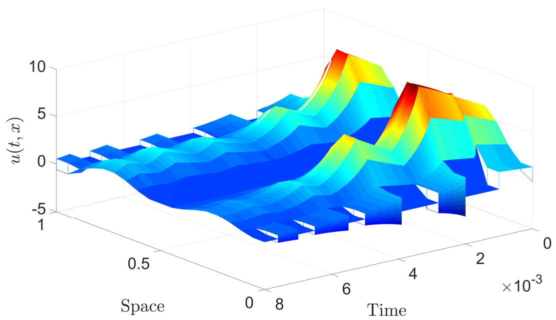

Fig. 2: Closed-loop response.

decays on I_1 (resp. I_2) and increases on I_2 (resp. I_1). We also observe the existence of a parameter-adaptation phase, since the decrease of V_1 is observed only starting from $[2, 2.8)$.

VI. Conclusion

We studied stabilization of the origin for the nonlinear KS equation subject to intermittent sensing when the coefficient λ_1 is unknown. Adaptive boundary controllers are designed to achieve GES in the L^2 sense. In future work, we would like to obtain robustness results for the perturbed equation, under the proposed sensing scenario, by guaranteeing input-to-state stability (ISS). Furthermore, solving the same problem using active control at $x = 0$ and $x = Y$ and null boundary conditions at $x = 1$, while guaranteeing input boundedness, is open, to the best of our knowledge.

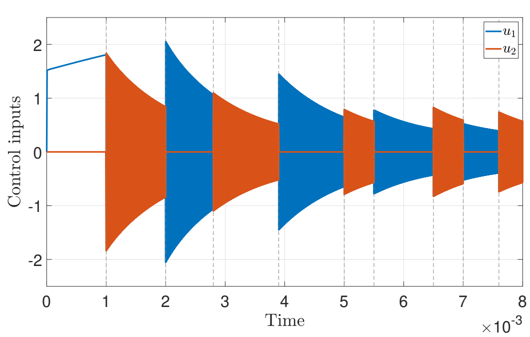

Fig. 3: The control inputs.

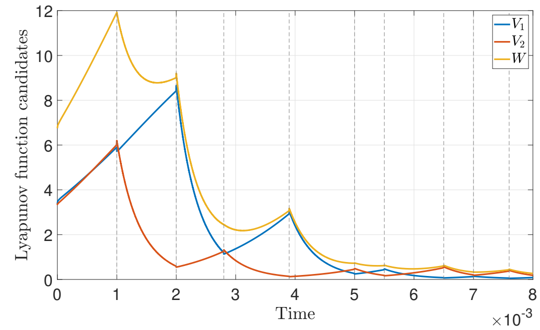

Fig. 4: Lyapunov function candidates.

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