Boundary Stabilization of Quasilinear Parabolic PDEs that Blow Up in Open Loop for Arbitrarily Small Initial Conditions

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Quasilinear parabolic PDEs that exhibit finite-time blow up model catastrophic events

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- Reactor core meltdowns
- Uncontrolled chain reactions
- ... Destruction of the plant

Benchmark Model

$$\Sigma : \begin{cases} u_t = \varepsilon(u)u_{xx} + \sum_{i=1}^{n} \gamma_i u^i u_x + u^p, & (1) \\ u_x(0) = v_0(u(0)), \quad u_x(1) = v_1(u(1)), & (2) \\ u(x,0) = u_0(x) \in \mathcal{H}^{2+\beta}[0,1], \quad \beta \in (0,1), & (3) \end{cases}$$

where $x \in [0,1]$ denotes the space variable, $t \geq 0$ denotes the time variable, and $u \in \mathbb{R}$ denotes the state variable.

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- Solutions understood in the classical sense

Open loop ($u_x(0) = u_x(1) = 0$)

- u blows-up in finite time if there exists $T_m \in (0, +\infty)$ s.t. $\lim_{t \to T_m^-} |u(\cdot, t)|_\infty = +\infty$

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Moreover, the blow-up time T_m verifies

$$T_m \leq \left(\int_0^1 u_o(x)dx \right)^{-1} \rightarrow \text{Independent of } \varepsilon$$

Open loop ($u_x(0) = u_x(1) = 0$)

Indeed, integrating both sides of $u_t = \varepsilon u_{xx} + u^2$ in space, we get

$$\begin{aligned}\frac{d}{dt}\int_0^1 u(x,t)dx &= \varepsilon \int_0^1 u_{xx}(x,t)dx + \int_0^1 u(x,t)^2 dx \\ &= \varepsilon u_x(1,t) - \varepsilon u_x(0,t) + \int_0^1 u(x,t)^2 dx \\ &= \int_0^1 u(x,t)^2 dx.\end{aligned}$$

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Using Cauchy-Schwarz inequality, we conclude that

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As a result, if $\int_0^1 u_o(x)dx > 0$, then u blows-up in finite time since $\int_0^1 u(x,t)dx \leq |u(\cdot,t)|_\infty$

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Best stability guarantees → estimate of the region of attraction

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  • Positivity of solutions is not preserved: state may become negative even if the initial condition is positive. Problematic in applications where negative values of the state are physically meaningless (chemical autocatalysis)

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• The Cole-Hopf transform is specific to Burgers' equation

Previous works

- Lyapunov method [Karafyllis, Krstic; SIAM'2019]:

Previous works

- Lyapunov method [Karafyllis, Krstic; SIAM'2019]: Global stabilization of diffusion-reaction PDEs with polynomial reaction

$$u_t = \varepsilon u_{xx} + u - u^3$$

• The reaction term $+u - u^3$ is stabilizing when the state is large → No finite-time blow-up phenomena in open loop

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  • Because of the nonlocal terms: difficult to study H^1 norm of the state, and to prove well-posedness

Assumption in our work

The map $u \mapsto \varepsilon(u)$ is differentiable, and there exist $\underline{\varepsilon} > 0$ and a continuous non-decreasing map $\bar{\varepsilon} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\inf_{u \in \mathbb{R}} \{\varepsilon(u)\} \geq \underline{\varepsilon}$$

and

$$|\varepsilon'(u)| \leq \bar{\varepsilon}(s) \qquad \forall u \in [-s, s], \quad \forall s \geq 0.$$

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$\rightarrow$ the PDE Σ is *uniformly parabolic*

Controllers

We propose Neumann cubic feedback laws

$$u_x(0) = v_o(u(0)) := \lambda_o u(0) + \mu u(0)^3,$$
$$u_x(1) = v_1(u(1)) := -\lambda_1 u(1) - \mu u(1)^3,$$

where

$$\mu := \frac{1}{\underline{\varepsilon}}\left(\frac{|\gamma_1|}{2} + \sum_{i=2}^{n} \frac{|\gamma_i| M^{i-2}}{i+2}\right)$$

and

$$\lambda_1 := \frac{2(m_1 + k_1) + |\gamma_1|}{2\underline{\varepsilon}}, \ \lambda_o := \frac{2(m_o + k_o) + |\gamma_1|}{2\underline{\varepsilon}}.$$

Here, $M, m_1, m_o > 0$ and $k_1, k_o \geq 0$ are control gains.

Stability

There exists $\omega > 0$, s.t., if

$$\kappa_o(u_o) := \left[|u_o|_{H^1}^2 + \lambda_1 u_o(1)^2 + \frac{\mu}{2} u_o(1)^4 + \lambda_o u_o(0)^2 + \frac{\mu}{2} u_o(0)^4 \right]^{1/2} \leq \sqrt{2\omega},$$

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then there exist $\sigma(\kappa_o) > 0$ and $\bar{\zeta} : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ continuous, nondecreasing in both arguments, and $\bar{\zeta}(0,0) = 0$, such that every complete solution satisfies, for all $t \geq 0$,

$$\begin{aligned}
|u(\cdot,t)|_{L^2}^2 &\leq |u_o|_{L^2}^2 \exp^{-\sigma t}, \\
|u(\cdot,t)|_{\infty}^2 &\leq 2\kappa_o |u_o|_{L^2} \exp^{-(\sigma/2)t} + |u_o|_{L^2}^2 \exp^{-\sigma t}, \\
|u_x(\cdot,t)|_{L^2}^2 &\leq \bar{\zeta}(\kappa_o, |u_o|_{L^2}) \exp^{-(\sigma/3)t} - \lambda_1 u(1,t)^2 \\
&\quad - \lambda_o u(0,t)^2 - \frac{\mu}{2} u(1,t)^4 - \frac{\mu}{2} u(0,t)^4.
\end{aligned}$$

Convergence

Moreover,

$$\lim_{t\to+\infty} \exp^{(\sigma/4)t} \max\left\{|u_{xx}(\cdot,t)|_{L^2}^2, |u_t(\cdot,t)|_{L^2}^2, |u_x(\cdot,t)|_{\infty}^2\right\} = 0.$$

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Moreover,

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No restrictions on the size of $|u_o''|_{L^2}$, $|u_t(\cdot,0)|_{L^2}$, and $|u_o'|_\infty$.

Region of attraction

$\omega > 0$ is any constant that verifies

$$m_1 + m_o > 2(6\omega)^{(p-1)/2} + (6\omega)^{p-1}/\underline{\varepsilon},$$

$$\underline{\varepsilon} - 2(m_1 + m_o) > \sqrt{6\omega}\bar{\varepsilon}(\sqrt{6\omega}) + \frac{n}{2\underline{\varepsilon}}\left(\sum_{i=1}^{n} \gamma_i^2 (6\omega)^i\right),$$

$$\sum_{i=2}^{n} \frac{|\gamma_i|}{i+2} M^{i-2} > \sum_{i=2}^{n} \frac{|\gamma_i|}{i+2} (6\omega)^{(i-2)/2} \text{ if } \exists i \in \{2, 3, ..., n\} \text{ s.t. } \gamma_i \neq 0.$$

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$\rightarrow$ At initial time,

$$\frac{\text{Reaction}}{\text{Diffusion}} < 1, \quad \frac{\text{Convection}}{\text{Diffusion}} < 1.$$

Decay rate

$$\sigma := m_1 + m_o - 2\left(\sqrt{3}\kappa_o\right)^{(p-1)} > 0,$$

$$\bar{\zeta} := \kappa_o^2 + 4(\lambda_1+\lambda_o)\kappa_o|u_o|_{L^2} + \frac{\mu}{5}|u_o|_{L^2}^4 + \frac{8\mu\kappa_o}{7}|u_o|_{L^2}^3 +$$

$$\left[\frac{\lambda_1+\lambda_o+4\mu\kappa_o^2}{2} + \frac{3^p\kappa_o^{2p-2}}{2\underline{\varepsilon}\big(m_1+m_o-2(6\omega)^{(p-1)/2}\big)}\right]|u_o|_{L^2}^2$$

$$+ \frac{(3/4)\left[\left(\sum_{i=1}^n \frac{n3^i\gamma_i^2\kappa_o^{2i}}{\underline{\varepsilon}}\right) + \frac{m_1+m_o}{3}\right]}{\underline{\varepsilon} - \sqrt{6\omega}\bar{\varepsilon}\left(\sqrt{6\omega}\right) - 2(m_o+m_1)}|u_o|_{L^2}^2.$$

Well-posedness

If $\varepsilon \in \mathcal{C}^2$, $\kappa_o(u_o) \leq \sqrt{2\omega}$, and the following compatibility condition holds

$$u_o'(0) = v_o(u_o(0)), \quad u_o'(1) = v_1(u_o(1)),$$

then there exists a unique solution that is complete, and any other solution must be a restriction of the complete solution.

Positivity of solutions

If $u_o(x) \geq 0$ for all $x \in [0,1]$ and u is a complete solution to Σ, then $u(x,t) \geq 0$ for all $(x,t) \in [0,1] \times [0,+\infty)$

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- Necessary to establish this property when the PDE's state is a physical nonnegative quantity: chemical concentration, temperature, ...

Invertibility of the feedback

The maps v_o and v_1 are invertible, i.e., there exist d_o, $d_1 : \mathbb{R} \to \mathbb{R}$ such that, given any solution u to Σ,

$$u_x(0) = v_o(u(0)) \iff u(0) = d_o(u_x(0)),$$
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In particular, when $\mu \neq 0$, with

$$d_l(y) := \sqrt[3]{\frac{(-1)^l y}{2\lambda_l} + \sqrt{\frac{y^2}{4\lambda_l^2} + \frac{\lambda_l^3}{27\mu^3}}} + \sqrt[3]{\frac{(-1)^l y}{2\lambda_l} - \sqrt{\frac{y^2}{4\lambda_l^2} + \frac{\lambda_l^3}{27\mu^3}}} \quad \forall l \in \{0, 1\},$$

and, when $\mu = 0$, with $d_l(y) := (-1)^l y/\lambda_l$ for all $l \in \{0, 1\}$.

Further results: One-sided boundary feedback

Suppose that $\gamma_k := 0$ for all $k \in \{1, 3, ...\}$, and $\gamma_k \geq 0$ (resp., $\gamma_k \leq 0$) for all $k \in \{2, 4, ...\}$.

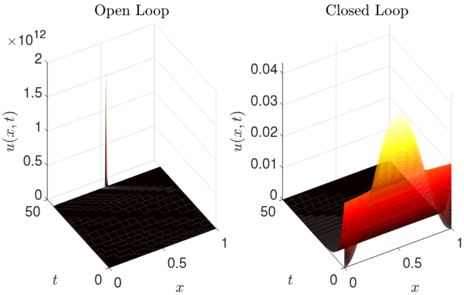
Further results: One-sided boundary feedback

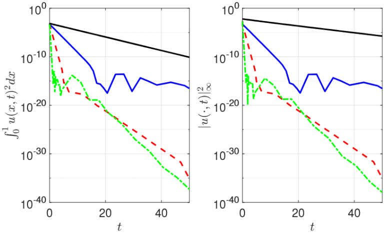
Suppose that $\gamma_k := 0$ for all $k \in \{1, 3, ...\}$, and $\gamma_k \geq 0$ (resp., $\gamma_k \leq 0$) for all $k \in \{2, 4, ...\}$.

Then, by designing v_1 (resp. v_o) as a Neumann cubic feedback while setting $v_o := 0$ (resp., $v_1 := 0$), the above stability, convergence, well-posedness and positivity guarantees remain verified.

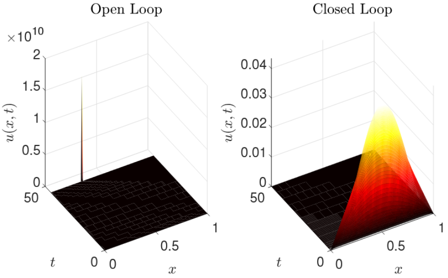
Simulation results

No convection





With convection



Research perspectives

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- Extension to PDEs defined on higher-dimensional domains

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- In-domain pointwise measurement and/or actuation