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State Estimation for the Kuramoto-Sivashinsky Equation Using Scanning Outputs

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***Abstract*— We study state estimation for the nonhomogeneous Kuramoto-Sivashinsky (KS) equation whose output takes location at a time-dependent position that browses the spatial domain back and forth. The only available data are the system's state and some of its spatial derivatives at the output's location. In this context, we construct a state observer by combining two KS equations. The first one is defined from one boundary up to the output's location, and the second one is defined from the output's location up to the other boundary. We design the observer's boundary conditions at the output's location to make the observation error converge, in the L^2 norm, to a ball centered at the origin of radius proportional to the size of the exogenous term affecting the KS equation. We show that this radius can be made arbitrarily small by appropriate tuning. Numerical simulations are performed to illustrate our results.**

***Index Terms*— Distributed parameter systems, parabolic equations, observer design, mobile outputs.**

I. Introduction

Reconstructing the profile of spatiotemporal entities, governed by Partial Differential Equations (PDE)s, is crucial in various contexts including environment monitoring, hydrology, meteorology [23], in addition to identification and control engineering [15, Chapter 5], [5]. Usually, only limited measurements are collected at each instant, since the sensors have a limited sensing range and due to the deployment of only few of them. This motivates the need for advanced estimation algorithms to handle practical scenarios of limited sensing.

For some specific classes of PDEs, boundary outputs are shown to be enough to reconstruct the state using a state observer. Results in this direction treated, among others, linear parabolic equations [21], hyperbolic equations [25], the nonlinear viscous Burgers equation [14], and the nonlinear heat equation [18]. However, there are still important classes of PDEs, including the KS equation in its general forms, for which we do not know yet whether boundary outputs are enough to design a converging state observer. Positive answers are provided only when the observer is tailored to a specific boundary design for the original equation [16]. Otherwise, an observer is designed in [10] for the linearized equation while using a weighted space integral of the state at discrete time instants as output. A relatively more general context of nonlinear KS equations with free boundary conditions is considered in [20], while assuming that the state is bounded and measured at a finite number of fixed output locations. As a result, the proposed observer is shown to converge exponentially, provided that the number of output locations is sufficiently large relatively to the size of the initial error.

The use of scanning and mobile outputs for such complex equations depicts well-motivated scenarios of limited sensing resources, in which, we might succeed to solve the estimation problem in situations where boundary outputs are not enough. For instance, in [6], optimal output-location trajectories are designed to guarantee state estimation for linear parabolic equations. An extension to linear wave equations is proposed in [7]. The problem of state estimation for the nonlinear heat equation is also considered in [26] via designing the output-location's motion. To the best of our knowledge, the state-estimation problem for the KS equation using scanning or mobile outputs has not yet been explored in the literature.

In this paper, we consider the nonlinear KS equation

$$u_t + uu_x + \lambda u_{xx} + u_{xxxx} = f(x,t), \tag{1}$$

where $x \in (0, L)$ is the space variable, $u \in \mathbb{R}$ is the state, $\lambda > 0$ is the *anti-diffusion* coefficient, otherwise known as the *destabilizing* coefficient, and $(x,t) \mapsto f(x,t)$ is an unknown exogenous term accounting for external perturbations or un-modeled dynamics. Equation (1) is subject to the Dirichlet-type boundary conditions:

$$\begin{cases} u(0,t) = h_1(t), \quad u(L,t) = h_2(t), \\ u_x(0,t) = h_3(t), \quad u_x(L,t) = h_4(t), \end{cases} \quad \text{for a.a. } t > 0, \tag{2}$$

where h_1, h_2, h_3, and h_4 are known time-dependent functions. The considered mobile output takes location at a time-dependent position $Y \in [0, L]$, which browses the spatial domain $[\varepsilon, L-\varepsilon]$ back and forth, for some $\varepsilon \geq 0$. As a result, the available measurements at time t are $u(x,t)$ for all $x \in [Y(t)-\varepsilon, Y(t)+\varepsilon]$ and the first- and third-order spatial derivatives $u_x(Y(t),t)$ and $u_{xxx}(Y(t),t)$. In particular, when $\varepsilon = 0$, only pointwise measurements are gathered. The proposed observer consists of two KS equations, one defined on $(0, Y(t))$ (provided that $Y \neq 0$) and the other one defined on $(Y(t), L)$ (provided that $Y(t) \neq L$). Besides copying the KS dynamics, the observers include boundary and in-domain terms that involve the collected measurements. As a result, we are able to ensure that the observation error converges, in the L^2 norm, to a ball centered at the origin of radius proportional to the size of f. This radius can be made arbitrarily small by tuning the observer's gains. We emphasis that we do not restrict the size of λ, L, and f nor the size of u. However, the coefficient λ is assumed to be known, and the spatial L^2 norm of the unknown function f is assumed to admit a known upperbound.

The rest of the paper is organized as follows. In Section II, the proposed sensing scenario is described, and the system of PDEs constituting the proposed observer is presented. In Section III, we design the observer boundary inputs and state our main result. In Section IV, we prove our main result. Finally, in Section V, we illustrate the main result by performing a set of numerical simulations.

Notation. For a function $u : [a,b] \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, $a \leq b$, its partial derivative with respect to t is denoted by u_t, its first-, second-, third- and fourth-order partial derivatives with respect to x are denoted by u_x, u_{xx}, u_{xxx}, and u_{xxxx}, respectively. For brievty, we may write $u(x)$ to mean the map $t \mapsto u(x,t)$, $u(t)$ to mean the map $x \mapsto u(x,t)$, and $u([a,b],t)$ to mean the set $\{u(x,t) : x \in [a,b]\}$. The time-derivative of a function $t \mapsto Y(t)$ is denoted either by

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$\dot{Y}(t)$ or by $\frac{d}{dt}Y(t)$. We denote by $L^2(a,b)$ the space of functions $v:[a,b]\to\mathbb{R}$ such that

$$|v|^2_{L^2(a,b)} := \int_a^b v(x)^2\ dx < +\infty.$$

We let $\mathcal{C}[a,b]$ be the space of continuous functions $v:[a,b]\to\mathbb{R}$. We denote by $H^n(a,b)$, for $n\in\{1,2,...\}$, the Sobolev space of functions $v\in L^2(a,b)$ such that v_x, v_{xx}, until the n^{th} order derivative of v belong to $L^2(a,b)$. Given $v\in H^4(a,b)$, we write

$$|v|^2_{H^4(a,b)} := |v|^2_{L^2(a,b)} + |v_x|^2_{L^2(a,b)} + ... + |v_{xxxx}|^2_{L^2(a,b)}.$$

Given a set X and $v:X\to\mathbb{R}$, we let

$$\operatorname*{ess\,sup}_{x\in X}|v(x)| := \inf_{c\in\mathbb{R}_{\geq 0}}\{|v(x)|\leq c \quad \text{almost everywhere on } X\}.$$

We denote by $H^{n+s}(a,b)$, for $n\in\{1,2,...\}$ and $s\in(0,1)$, the fractional Sobolev space of functions $v\in H^n(a,b)$ such that

$$\int_a^b\int_a^b \frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}}dxdy < +\infty.$$

We denote by $L^2_{loc}(0,+\infty;H^4(a,b))$ the space of functions $v:[0,+\infty)\times[a,b]\to\mathbb{R}$ such that $v(t)\in H^4(a,b)$ for a.a. $t\in(0,+\infty)$ and $\int_I |v(t)|^2_{L^2(a,b)}dt < +\infty$ for any compact set $I\subset[0,+\infty)$. Finally, a.a. stands for almost all and a.e. stands for almost everywhere.

II. Problem Statement

We start recalling that a solution to (1)-(2) starting from the initial condition $u_o\in L^2(0,L)$ is a function $u\in L^2_{loc}(0,+\infty;H^4(0,L))$ with $u_t\in L^2_{loc}(0,+\infty;L^2(0,L))$, verifying (1) for a.a. $x\in(0,L)$ and for a.a. $t>0$, verifying (2) for a.a. $t>0$, and such that $u(x,0)=u_o(x)$ for a.a. $x\in[0,L]$.

Well-posedness for (1)-(2) is guaranteed only in specific situations. For example, it is shown in [17] that, when $u_o\in H^4(0,L)$, $f\equiv 0$,

$$(h_1,h_2,h_3,h_4)\in\left(H^{1+3/8}(0,+\infty)\right)^2\times\left(H^{1+1/8}(0,+\infty)\right)^2,$$

and provided that the following compatibility condition holds

$$u_o(0)=h_1(0),\quad u_o(L)=h_2(0),$$
$$u_{ox}(0)=h_3(0),\quad u_{ox}(L)=h_4(0),$$

there exists a unique solution to (1)-(2) starting from u_o. Moreover, the map $t\mapsto|u(t)|_{H^4(0,L)}$ is continuous on $[0,+\infty)$.

Throughout this study, we assume that (1)-(2) admits solutions defined on $[0,+\infty)$, according to the concept above.

A. The Scanning Output

Given $\varepsilon\in[0,L/2]$ and a strictly-increasing sequence $\{T_k\}_{k=0}^{\infty}\subset\mathbb{R}_{\geq 0}$ with $T_0:=0$, the output location is defined by a continuously differentiable function $Y:\mathbb{R}_{\geq 0}\to[\varepsilon,L-\varepsilon]$ such that

S1 - $Y(T_{2k})=\varepsilon$ for all $k\in\mathbb{N}$.

S2 - $Y(T_{2k+1})=L-\varepsilon$ for all $k\in\mathbb{N}$.

Furthermore, given a solution u to (1)-(2), the variables

$$u([Y(t)-\varepsilon,Y(t)+\varepsilon],t),\quad u_x(Y(t),t),\quad u_{xxx}(Y(t),t) \tag{3}$$

are available for a.a. $t>0$.

Remark 2.1: When $\varepsilon>0$, we conclude that the output location will not browse the spatial domain all the way to its extremities. However, when $\varepsilon=0$, the output location browses the entire spatial domain, while collecting only pointwise measurements. In other words, the constraints on the sensor's trajectory differ between the two cases $\varepsilon>0$ and $\varepsilon=0$. Furthermore, it is worth emphasizing that in many applications, the available sensing devices provide distributed measurements over some spatial range, rather than a measurement at a single point (pointwise measurement); these applications include–but are not restricted to–thermography sensors for temperature measurement [24], imaging using cameras mounted on unmanned aerial vehicles [11], scanning tunneling microscopes for surface imaging [3], and scanning thermal conductivity microscopes for producing thermal conductivity maps with submicron resolution [8]. •

Example 2.1: A candidate function $t\mapsto Y(t)$ verifying S1-S2 is

$$Y(t):=(L-2\varepsilon)\sin(\omega t)^2+\varepsilon,\quad \omega>0. \tag{4}$$

In this case, $Y(t)=\varepsilon$ if and only if $t\in\{k\pi/\omega:k\in\mathbb{N}\}$. Thus, we would have $T_{2k}:=k\pi/\omega$ for all $k\in\mathbb{N}$.

Similarly, $Y(t)=L-\varepsilon$ if and only if $t\in\{(2k+1)\frac{\pi}{2\omega}:k\in\mathbb{N}\}$. Thus, we would have

$$T_{2k+1}:=(2k+1)\pi/(2\omega)\quad\forall k\in\mathbb{N}. \tag{5}$$

Remark 2.2: In practical situations, if the sensor cannot directly measure $u_x(Y(t),t)$ and $u_{xxx}(Y(t),t)$, these values can be approximated through collecting four pointwise measurements $u(Y(t))$, $u(Y(t)+\Delta)$, $u(Y(t)+2\Delta)$, and $u(Y(t)+3\Delta)$ for some $\Delta\in\mathbb{R}$ sufficiently small, in which case, we would have

$$u_x(Y(t))\approx\frac{u(Y(t)+\Delta)-u(Y(t))}{\Delta},$$
$$u_{xxx}(Y(t))\approx \frac{u(Y(t)+3\Delta)-3u(Y(t)+2\Delta)+3u(Y(t)+\Delta)-u(Y(t))}{\Delta^3}.$$

This may suggest the use of four moving sensors, or replacing the value of u at each location by the most recent value collected at that location. •

B. Observer Design

The observer dynamics are governed by

$$\begin{cases}\hat{w}_t+\hat{w}\hat{w}_x+\lambda\hat{w}_{xx}+\hat{w}_{xxxx}+2\delta_Y\zeta_1=0 & x\in(0,Y)\\ \hat{v}_t+\hat{v}\hat{v}_x+\lambda\hat{v}_{xx}+\hat{v}_{xxxx}+2\delta_Y\zeta_2=0 & x\in(Y,L),\end{cases} \tag{6}$$

which is subject to the boundary conditions

$$\begin{cases}\hat{w}(0)=h_1,\quad \hat{w}_x(0)=h_3,\\ \hat{w}(Y)=u(Y)-\zeta_3,\quad \hat{w}_x(Y)=u_x(Y)\quad\text{when } Y\neq 0,\end{cases} \tag{7}$$

$$\begin{cases}\hat{v}(L)=h_2,\quad \hat{v}_x(L)=h_4,\\ \hat{v}(Y)=u(Y)-\zeta_3,\quad \hat{v}_x(Y)=u_x(Y)\quad\text{when } Y\neq L.\end{cases} \tag{8}$$

Here, $\delta_Y(x):=\delta(x-Y)$ for all $x\in[0,L]$ with δ being the Dirac distribution, which verifies

$$\int_0^Y \hat{w}(x)\delta_Y(x)dx=\frac{1}{2}\hat{w}(Y),\quad \int_Y^L\hat{v}(x)\delta_Y(x)dx=\frac{1}{2}\hat{v}(Y).$$

The latter two equations are obtained from [1, Equation (5)] while replacing therein u by $\hat{w}$ and $\hat{v}$, D by $(0,Y)$ and (Y,L), Γ by $\{0,Y\}$ and $\{Y,L\}$, respectively, and y by Y.

The time-dependent in-domain terms ζ_1 and ζ_2 are designed as

$$\zeta_1:=-\frac{4}{3}\zeta_3^2+\tilde{w}_{xxx}(Y)-\frac{1}{2}\big(\dot{Y}-2u(Y)\big)\zeta_3 \quad \text{a.e. on } [0,+\infty) \text{ when } Y\neq 0, \tag{9}$$

$$\zeta_2:=\frac{4}{3}\zeta_3^2-\tilde{v}_{xxx}(Y)+\frac{1}{2}\big(\dot{Y}-2u(Y)\big)\zeta_3 \quad \text{a.e. on } [0,+\infty) \text{ when } Y\neq L, \tag{10}$$

where

$$\tilde{w} := u - \hat{w} \quad \text{on } [0, Y], \quad \tilde{v} := u - \hat{v} \quad \text{on } [Y, L].$$

Furthermore, the time-dependent boundary term ζ_3 is designed as

$$\zeta_3 := \begin{cases} -(\vartheta_1 + \alpha)\bar{V}_1 & \text{a.e. on } [0, T_1), \\ -(\vartheta_1 + \alpha)\bar{V}_1 - (\vartheta_2 + \alpha)\bar{V}_2 & \text{a.e. on } [T_1, +\infty), \end{cases} \tag{11}$$

where

$$\vartheta_1(t) := \operatorname*{ess\,sup}_{x \in [0, Y(t)]} |\hat{w}(x,t)|^2 + \theta_1(t) + \frac{\sigma\sqrt{3}}{2}, \tag{12}$$

$$\vartheta_2(t) := \operatorname*{ess\,sup}_{x \in [Y(t), L]} |\hat{v}(x,t)|^2 + \theta_2(t) + \frac{\sigma\sqrt{3}}{2}, \tag{13}$$

$$\begin{aligned} \bar{V}_1(t) := &\int_{T_{2k}}^{t} \left(\vartheta_1(s)\bar{V}_1(s) + \zeta_3(s) + \frac{\sqrt{3}}{2\sigma}\bar{f}^2 \right) ds \\ &+ V_1(T_{2k}) \qquad \forall t \in [T_{2k}, T_{2k+2}), \ \forall k \in \mathbb{N}, \end{aligned} \tag{14}$$

$$\begin{aligned} \bar{V}_2(t) := &\int_{T_{2k+1}}^{t} \left(\vartheta_2(s)\bar{V}_2(s) + \zeta_3(s) + \frac{\sqrt{3}}{2\sigma}\bar{f}^2 \right) ds \\ &+ V_2(T_{2k+1}) \qquad \forall t \in [T_{2k+1}, T_{2k+3}), \ \forall k \in \mathbb{N}, \end{aligned} \tag{15}$$

and

$$V_1(T_{2k}) := \frac{3}{4}\int_0^{Y(T_{2k})=\varepsilon} \tilde{w}(x, T_{2k})^2 dx,$$

$$V_2(T_{2k+1}) := \frac{3}{4}\int_{Y(T_{2k+1})=(L-\varepsilon)}^{L} \tilde{v}(x, T_{2k+1})^2 dx.$$

In (14) and (15), the constant $\bar{f}$ is an upperbound on the function $t \mapsto |f(t)|_{L^2(0,L)}$, i.e.,

$$\operatorname*{ess\,sup}_{t>0} \int_0^L f(x,t)^2 dx \leq \bar{f}^2. \tag{16}$$

Finally, $\alpha, \sigma > 0$ are free design parameters, and

$$\theta_1 := \begin{cases} 0 & \text{if } Y \leq \frac{\pi}{\sqrt{\lambda + \frac{1}{2}}} \\ (2\lambda + 1)\left[\lambda + \frac{1}{2} + \frac{12}{Y^2}\right] & \text{if } Y > \frac{\pi}{\sqrt{\lambda + \frac{1}{2}}}, \end{cases} \tag{17}$$

$$\theta_2 := \begin{cases} 0 & \text{if } L - Y \leq \frac{\pi}{\sqrt{\lambda + \frac{1}{2}}} \\ (2\lambda + 1)\left[\lambda + \frac{1}{2} + \frac{12}{(L-Y)^2}\right] & \text{if } L - Y > \frac{\pi}{\sqrt{\lambda + \frac{1}{2}}}. \end{cases} \tag{18}$$

The rationale behind the proposed in-domain and boundary designs is justified in Remarks 3.1, 3.2, and 3.3, after analyzing the variations of some Lyapunov functions along the dynamics of $(\tilde{w}, \tilde{v})$. However, to compute such variations, we need to specify the concept of solutions to (6)-(7)-(8) under (9)-(10) and (11).

Definition 2.1 (Concept of solutions): Given a solution u to (1)-(2), the pair $(\hat{w}, \hat{v})$ is a solution to (6)-(7)-(8) under (9)-(10) and (11) starting from $(\hat{w}_o, \hat{v}_o) \in L^2(0, Y(0)) \times L^2(Y(0), L)$, if the following properties hold.

D1- $\hat{w}(\cdot, t) \in H^4(0, Y(t))$ for a.a. $t > 0$ such that $Y(t) \neq 0$, and $\hat{v}(\cdot, t) \in H^4(Y(t), L)$ for a.a. $t > 0$ such that $Y(t) \neq L$.

D2- $\hat{w}_t \in L^2(0, Y(t))$ and $\hat{v}_t \in L^2(Y(t), L)$ for a.a. $t > 0$.

D3- For any bounded set $I \subset [0, +\infty)$, the following two inequalities hold

$$\max\left\{ \int_I |\hat{w}(t)|^2_{L^2(0,Y(t))} dt, \ \int_I |\hat{v}(t)|^2_{L^2(Y(t),L)} dt \right\} < \infty,$$

$$\max\left\{ \int_I |\hat{w}_t(t)|^2_{L^2(0,Y(t))} dt, \ \int_I |\hat{v}_t(t)|^2_{L^2(Y(t),L)} dt \right\} < \infty.$$

D4- For any bounded set $I \subset [0, +\infty)$, the following two inequalities hold

$$\int_I |\hat{w}_x(t)|^2_{L^2(0,Y(t))} dt < +\infty \quad \text{if } Y(t) \neq 0 \text{ for all } t \in I,$$

$$\int_I |\hat{v}_x(t)|^2_{L^2(Y(t),L)} dt < +\infty \quad \text{if } Y(t) \neq L \text{ for all } t \in I.$$

D5- The boundary conditions in (7) and (8) are verified for a.a. $t > 0$. Furthermore, a.e. on $[0, +\infty)$ when $Y \neq 0$,

$$\begin{aligned} \int_0^Y \Big(\hat{w}_t(x) + \hat{w}(x)\hat{w}_x(x) + \lambda\hat{w}_{xx}(x) + \hat{w}_{xxxx}(x) \Big)\varphi(x) dx \\ = -\zeta_1\varphi(Y) \quad \forall \varphi \in \mathcal{C}[0, Y]. \end{aligned} \tag{19}$$

Similarly, a.e. on $[0, +\infty)$ when $Y \neq L$,

$$\begin{aligned} \int_Y^L \Big(\hat{v}_t(x) + \hat{v}(x)\hat{v}_x(x) + \lambda\hat{v}_{xx}(x) + \hat{v}_{xxxx}(x) \Big)\varphi(x) dx \\ = -\zeta_2\varphi(Y) \quad \forall \varphi \in \mathcal{C}[Y, L]. \end{aligned} \tag{20}$$

D6- $\hat{w}(x, 0) = \hat{w}_o(x)$ for a.a. $x \in [0, Y(0)]$ and $\hat{v}(x, 0) = \hat{v}_o(x)$ for a.a. $x \in [Y(0), L]$.

•

We believe that proving the existence of solutions for the proposed observer is out of reach. In particular, the Delta distribution in (6), which is not locally integrable in x, and the boundary term ζ_3, involving norms of the state and its delayed versions, make the fixed-point type arguments not applicable, to the best of our knowledge. Hence, we assume throughout this paper that the observer's solutions exist and are defined on $[0, +\infty)$.

Remark 2.3: The presence in (6) of the Delta distribution, which is not locally integrable in x, justifies the considered concept of solutions. Such solutions are referred to as *weak* solutions [4]. We note that the Dirac distribution has also been used in [7] to design a state observer for the linear wave equation. •

In the following proposition, we show that, under the proposed concept of solutions, ζ_1, ζ_2, and ζ_3 are well defined. The proof is in the Appendix.

Proposition 1: Consider $u \in L^2_{loc}(0, +\infty; H^4(0, L))$ and a pair $(\hat{w}, \hat{v})$ verifying item D1, the first inequality of item D3 for any bounded set $I \subset [0, +\infty)$, and item D4 of Definition 2.1. Then, the functions ζ_1, ζ_2, and ζ_3, in (9), (10), and (11), respectively, verify

$$\zeta_3(t) < +\infty \text{ for a.a. } t \in [0, +\infty), \tag{21}$$

$$\zeta_1(t) < +\infty \text{ for a.a. } t \in [0, +\infty) \colon Y(t) \neq 0, \tag{22}$$

$$\zeta_2(t) < +\infty \text{ for a.a. } t \in [0, +\infty) \colon Y(t) \neq L. \tag{23}$$

□

III. Main Result

Before presenting our main result, we introduce the state estimate

$$\hat{u}(x,t) := \begin{cases} \hat{w}(x,t) & \forall (x,t) \in [0, Y] \times [0, +\infty), \\ \hat{v}(x,t) & \forall (x,t) \in (Y, L] \times [0, +\infty). \end{cases} \tag{24}$$

Furthermore, we introduce the estimation-error variable

$$\tilde{u} := u - \hat{u}.$$

Theorem 1: Assume that a solution u to (1)-(2) exists and is defined on $[0, +\infty)$. Let $\varepsilon \in [0, L/2]$, a continuously differentiable function $Y : \mathbb{R}_{\geq 0} \to [\varepsilon, L - \varepsilon]$, and a strictly increasing sequence $\{T_k\}_{k=0}^{\infty} \subset \mathbb{R}_{\geq 0}$, with $T_0 = 0$, such that S1-S2 in Section II-A hold and the variables in (3) are available for a.a. $t > 0$. Then, provided that a solution $(\hat{w}, \hat{v})$ to (6)-(7)-(8) under (9)-(10) and (11) exists and is defined on $[0, +\infty)$, we conclude that

$$|\tilde{u}(t)|_{L^2(0,L)} \leq |\tilde{u}(T_1)|_{L^2(0,L)} \exp^{-\frac{\alpha}{2}(t-T_1)} + \left(\frac{\sqrt{2\left(1-\exp^{-\alpha(t-T_1)}\right)}}{3^{1/4}\sqrt{\alpha\sigma}} \right) \bar{f} \qquad \forall t \geq T_1, \tag{25}$$

where α, $\sigma > 0$ are the observer's free design parameters, and $\bar{f}$ is an upperbound of the function $t \mapsto |f(t)|_{L^2(0,L)}$ verifying (16). □

The proof of Theorem 1 is based on the following key lemmas. The first one derives integral equations governing the evolution of $(\tilde{w}, \tilde{v})$, which allow the analysis of the behavior of $\tilde{u}$.

Lemma 1: Let u be a solution to (1)-(2) and $(\hat{w}, \hat{v})$ be a solution to (6)-(7)-(8) under (9)-(10) and (11). Then, a.e. on $(0, +\infty)$ (when $Y \neq 0$), we have

$$\begin{aligned} &\int_0^Y \Big(\tilde{w}_t(x) + \tilde{w}(x)\tilde{w}_x(x) + \hat{w}(x)\tilde{w}_x(x) + \hat{w}_x(x)\tilde{w}(x) \\ &\qquad + \lambda \tilde{w}_{xx}(x) + \tilde{w}_{xxxx}(x) \Big) \varphi(x)dx \\ &\qquad = \zeta_1 \varphi(Y) + \int_0^Y f(x)\varphi(x)dx \quad \forall \varphi \in \mathcal{C}[0, Y], \end{aligned} \tag{26}$$

$$\tilde{w}(0) = \tilde{w}_x(0) = 0, \quad \tilde{w}_x(Y) = 0, \quad \tilde{w}(Y) = \zeta_3. \tag{27}$$

Similarly, a.e. on $(0, +\infty)$ (when $Y \neq L$), we have

$$\begin{aligned} &\int_Y^L \Big(\tilde{v}_t(x) + \tilde{v}(x)\tilde{v}_x(x) + \hat{v}(x)\tilde{v}_x(x) + \hat{v}_x(x)\tilde{v}(x) \\ &\qquad + \lambda \tilde{v}_{xx}(x) + \tilde{v}_{xxxx}(x) \Big) \varphi(x)dx \\ &\qquad = \zeta_2 \varphi(Y) + \int_Y^L f(x)\varphi(x)dx \quad \forall \varphi \in \mathcal{C}[Y, L], \end{aligned} \tag{28}$$

$$\tilde{v}(L) = \tilde{v}_x(L) = 0, \quad \tilde{v}_x(Y) = 0, \quad \tilde{v}(Y) = \zeta_3. \tag{29}$$

□

Proof: Since u is a solution to (1)-(2), then a.e. on $(0, +\infty)$ we have

$$\begin{aligned} &\int_0^Y \Big(u_t(x) + u(x)u_x(x) + \lambda u_{xx}(x) + u_{xxxx}(x) \Big) \varphi(x)dx \\ &\qquad = \int_0^Y f(x)\varphi(x)dx \quad \forall \varphi \in \mathcal{C}[0, Y]. \end{aligned} \tag{30}$$

Subtracting (19) from (30), we get

$$\begin{aligned} &\int_0^Y \Big(\tilde{w}_t(x) - \hat{w}(x)\hat{w}_x(x) + u(x)u_x(x) \\ &\qquad + \lambda \tilde{w}_{xx}(x) + \tilde{w}_{xxxx}(x) \Big) \varphi(x)dx \\ &\qquad = \zeta_1 \varphi(Y) + \int_0^Y f(x)\varphi(x)dx \quad \forall \varphi \in \mathcal{C}[0, Y]. \end{aligned} \tag{31}$$

Note that

$$\tilde{w}\tilde{w}_x = (u - \hat{w})(u_x - \hat{w}_x) = uu_x + \hat{w}\hat{w}_x - u\hat{w}_x - \hat{w}u_x, \tag{32}$$

$$\hat{w}\tilde{w}_x = \hat{w}(u_x - \hat{w}_x) = \hat{w}u_x - \hat{w}\hat{w}_x, \tag{33}$$

$$\hat{w}_x\tilde{w} = \hat{w}_x(u - \hat{w}) = \hat{w}_x u - \hat{w}_x\hat{w}. \tag{34}$$

By summing (32)-(34), we obtain

$$\tilde{w}\tilde{w}_x + \hat{w}\tilde{w}_x + \hat{w}_x\tilde{w} = uu_x - \hat{w}\hat{w}_x. \tag{35}$$

Finally, combining (31) and (35), we obtain the first equation in (26). We prove the second equation in (28) in a similar way. ■

Next, we introduce the Lyapunov functional candidates

$$V_1(\tilde{w}) := \frac{3}{4}\int_0^Y \tilde{w}(x)^2 dx, \quad V_2(\tilde{v}) := \frac{3}{4}\int_Y^L \tilde{v}(x)^2 dx. \tag{36}$$

The next Lemma, whose proof is in the Appendix, derives some key upperbounds on $(\dot{V}_1, \dot{V}_2)$ along the dynamics of $(\tilde{w}, \tilde{v})$.

Lemma 2: Let $\sigma > 0$, let u be a solution to (1)-(2), and let $(\hat{w}, \hat{v})$ be a solution to (6)-(7)-(8) under (9)-(10) and (11). Then, a.e. on $(0, +\infty)$, we have

$$\begin{aligned} \dot{V}_1 &\leq \vartheta_1 V_1 + \frac{\sqrt{3}}{2\sigma}\int_0^Y f(x)^2 dx + \frac{3}{2}\zeta_1\zeta_3 + \zeta_3^3 - \frac{3}{2}\tilde{w}_{xxx}(Y)\zeta_3 \\ &\qquad + \frac{3}{4}\big(\dot{Y} - 2u(Y)\big)\zeta_3^2 \quad \text{when } Y \neq 0, \end{aligned} \tag{37}$$

$$\begin{aligned} \dot{V}_2 &\leq \vartheta_2 V_2 + \frac{\sqrt{3}}{2\sigma}\int_Y^L f(x)^2 dx + \frac{3}{2}\zeta_2\zeta_3 - \zeta_3^3 + \frac{3}{2}\tilde{v}_{xxx}(Y)\zeta_3 \\ &\qquad - \frac{3}{4}\big(\dot{Y} - 2u(Y)\big)\zeta_3^2 \quad \text{when } Y \neq L, \end{aligned} \tag{38}$$

where the time-dependent variables ϑ_1 and ϑ_2 are introduced in (12) and (13), respectively. □

Before proving Theorem 1, the following remarks are in order.

Remark 3.1: According to (37) and due to the cubic term ζ_3^3 therein, one would want to select $\zeta_3 < 0$ with $|\zeta_3|$ large to force V_1 to decay towards a certain neighborhood of the origin whose radius depends on f. However, to achieve the same decay property for V_2, according to (38) and due to the cubic term $-\zeta_3^3$ therein, one would want to set $\zeta_3 > 0$ with $|\zeta_3|$ sufficiently large. Hence, forcing the decay of both V_1 and V_2 is not possible under this approach. To resolve this bottleneck, the proposed design of (ζ_1, ζ_2) is key, as it guarantees, a.e. on $(0, +\infty)$, that

$$\begin{aligned} \dot{V}_1 &\leq \vartheta_1 V_1 + \frac{\sqrt{3}}{2\sigma}\int_0^Y f(x)^2 dx + \zeta_3 \quad \text{when } Y \neq 0, \\ \dot{V}_2 &\leq \vartheta_2 V_2 + \frac{\sqrt{3}}{2\sigma}\int_Y^L f(x)^2 dx + \zeta_3 \quad \text{when } Y \neq L. \end{aligned} \tag{39}$$

•

Remark 3.2: If we set $\zeta_1 = \zeta_2 = \zeta_3 = 0$, which means that $\hat{w}(Y) = u(Y)$ whenever $Y \neq 0$ and $\hat{v}(Y) = u(Y)$ whenever $Y \neq L$, by Lemma 2, we conclude that, for a.a. time,

$$\begin{aligned} \dot{V}_1 &\leq \vartheta_1 V_1 + \frac{\sqrt{3}}{2\sigma}\int_0^Y f(x)^2 dx \quad \text{when } Y \neq 0, \\ \dot{V}_2 &\leq \vartheta_2 V_2 + \frac{\sqrt{3}}{2\sigma}\int_Y^L f(x)^2 dx \quad \text{when } Y \neq L. \end{aligned} \tag{40}$$

If we additionally assume that $\varepsilon = 0$, then

$$V_1(\tilde{w}(T_{2k})) = V_2(\tilde{v}(T_{2k+1})) = 0 \qquad \forall k \in \mathbb{N}.$$

This implies, when $f = 0$, that

$$V_1(\tilde{w}(t)) = V_2(\tilde{v}(t)) = 0 \qquad \forall t \geq T_1.$$

However, when $V_1(\tilde{w}(T_{2k})) \neq 0$ or $V_2(\tilde{v}(T_{2k+1})) \neq 0$, due to $\varepsilon > 0$, or when $\varepsilon = 0$ but $f \neq 0$, V_1 and V_2 are not prevented from growing unboundedly. This is mainly because ϑ_1 and ϑ_2 can grow unboundedly. Hence, the proposed non-trivial design of ζ_3 is key to guarantee robustness with respect to both the perturbation f and the output-position offset ε. •

Remark 3.3: Knowing the upper-bound $\bar{f}$ verifying (16), one can attempt to design ζ_3 so that, for some $\alpha, \beta > 0$,[1]

$$\zeta_3 \leq -(\vartheta_1 + \alpha)V_1 - (\vartheta_2 + \alpha)V_2 - \beta\bar{f}^2 \quad \text{a.e. on } [T_1, +\infty). \tag{41}$$

[1] As it appears from the proof of Theorem 1, it is, in fact, possible to enforce (41) using only the available measurements.

In this case, using (39), we guarantee that

$$\dot{V}_1 \leq -\alpha V_1 + \frac{\sqrt{3}}{2\sigma}\int_0^Y f(x)^2 dx - \beta \bar{f}^2 \quad \text{a.e. on } (T_1, +\infty), \text{ when } Y \neq 0.$$

Hence, if $\bar{f}$ is sufficiently large so that

$$\frac{\sqrt{3}}{2\sigma}\int_0^Y f(x)^2 dx - \beta \bar{f}^2 \leq -1 \qquad \text{a.e. on } (T_1, +\infty),$$

we conclude that

$$\dot{V}_1 \leq -\alpha V_1 - 1 \qquad \text{a.e. on } (T_1, +\infty), \text{ when } Y \neq 0. \tag{42}$$

Consequently, if, e.g., $Y(t) \neq 0$ for a.a. $t > 0$, then, since V_1 cannot be negative, (42) contradicts the assumption that the observer's solutions are defined on $[0, +\infty)$. As a result, under the choice of ζ_3 ensuring (41), the observer's solutions would not be defined on $[0, +\infty)$ if the upperbound $\bar{f}$ is excessive. •

IV. Proof of Theorem 1

Using the proposed design of (ζ_1, ζ_2) in (9)-(10), we conclude that, a.e. on $[0, +\infty)$,

$$\frac{3}{2}\zeta_1\zeta_3 = -2\zeta_3^3 - \frac{3}{4}\big(\dot{Y} - 2u(Y)\big)\zeta_3^2 + \frac{3}{2}\tilde{w}_{xxx}(Y)\zeta_3 \text{ when } Y \neq 0.$$
$$\frac{3}{2}\zeta_2\zeta_3 = 2\zeta_3^3 + \frac{3}{4}\big(\dot{Y} - 2u(Y)\big)\zeta_3^2 - \frac{3}{2}\tilde{v}_{xxx}(Y)\zeta_3 \quad \text{when } Y \neq L.$$

Hence, using (37) and (38), we verify the inequalities in (39) for a.a. $t > 0$. Now, based on (39), we can make both V_1 and V_2 decay towards a neighborhood of the origin of radius dependent on f by taking ζ_3 negative and sufficiently large in norm. In particular, we propose to show that our choice of ζ_3 in (11) verifies

$$\zeta_3 \leq -(\vartheta_1 + \alpha) V_1 - (\vartheta_2 + \alpha) V_2 \quad \text{a.e. on } [T_1, +\infty), \tag{43}$$

where $\alpha > 0$ is introduced in (11) and $T_1 > 0$ is introduced in Section II-A and corresponds to $\min\{T_i : Y(T_i) = L - \varepsilon\}$. Indeed, if (43) is verified, then in view of (39), we conclude that, a.e. on $[T_1, +\infty)$,

$$\dot{V}_1 \leq -\alpha V_1 + \frac{\sqrt{3}}{2\sigma}\int_0^Y f(x)^2 dx \qquad \text{when } Y \neq 0, \tag{44}$$

$$\dot{V}_2 \leq -\alpha V_2 + \frac{\sqrt{3}}{2\sigma}\int_Y^L f(x)^2 dx \qquad \text{when } Y \neq L. \tag{45}$$

On the other hand, on each open interval $I \subset [T_1, +\infty)$ such that $Y(t) = 0$ (resp., $Y(t) = L$) for all $t \in I_1$, we conclude that $V_1(t) = 0$ (resp., $V_2(t) = 0$) for all $t \in I$. Consequently, the inequality in (44) (resp., (45)) holds also a.e. on I. As a consequence, a.e. on $[T_1, +\infty)$, we verify

$$\dot{V}_1 + \dot{V}_2 \leq -\alpha(V_1 + V_2) + \frac{\sqrt{3}}{2\sigma}\bar{f}^2, \tag{46}$$

with $\bar{f}$ being the constant upperbound in (16). Integrating the latter inequality, we obtain

$$V_1(t) + V_2(t) \leq (V_1(T_1) + V_2(T_1)) \exp^{-\alpha(t - T_1)} + \frac{\sqrt{3}}{2\alpha\sigma}\left(1 - \exp^{-\alpha(t-T_1)}\right)\bar{f}^2 \quad \forall t \geq T_1. \tag{47}$$

The latter is enough to guarantee (25).

To complete the proof, we verify (43). To do so, it is enough to show that

$$V_1(t) \leq \bar{V}_1(t) \qquad \forall t \geq 0, \qquad V_2(t) \leq \bar{V}_2(t) \qquad \forall t \geq T_1.$$

We verify the latter using the classical comparison Lemma; see Lemma 3.4 in [12]. Indeed, for each $k \in \{0, 1, ...\}$, on the interval $[T_{2k}, T_{2k+2})$, any solution V_1 to the first inequality in (39), which verifies $V_1(t) = 0$ if $Y(t) = 0$, is upperbounded by the solution to

$$\dot{\bar{V}}_1 = \vartheta_1 \bar{V}_1 + \zeta_3 + \frac{\sqrt{3}}{2\sigma}\bar{f}^2 \text{ with } \bar{V}_1(T_{2k}) = V_1(T_{2k}). \tag{48}$$

Integrating the latter ODE, we find (14). The same reasoning applies to show that $V_2(t) \leq \bar{V}_2(t)$ for all $t \geq T_1$.

V. Simulation Results

In this section, we illustrate our main result with some numerical simulations performed on MATLAB. The numerical scheme is described in the Appendix of [2].

We consider the KS equation (1)-(2) with $\lambda = 4$, $L = 30$, and $h_1 = h_2 = h_3 = h_4 = 0$. The perturbation f is a Gaussian white noise generated using the MATLAB *wgn* function, for which, (16) holds with the upperbound $\bar{f} = 0.054$. The equation's initial condition is given by $u_o(x) := 5\exp^{-(x-15)^2}$ for all $x \in [0, L]$. We plot the response $(x, t) \mapsto u(x, t)$ of the KS equation in Figure 1 (left).

The moving output location $t \mapsto Y(t)$ is given by (4) with $\varepsilon := 2$ and $\omega := 2\pi$. In particular, according to (5), we have $T_1 = 0.25$. In Figure 1 (middle), we simulate the response of the observer (6)-(7)-(8) under (9)-(10) and (11), starting from the initial condition $\hat{w}_o(x) = 0$ for all $x \in [0, Y]$, $\hat{v}_o(x) := 0$ for all $x \in [Y, L]$, and using the observer's gains $(\alpha, \sigma) := (1, 100)$. In Figure 1 (right), we plot the observation error $(x, t) \mapsto \tilde{u}(x, t) := u(x, t) - \hat{u}(x, t)$. Furthermore, in Figure 2 (left), we plot the evolution in time of the L^2 norm of the observation error as well as its upperbound in (25). As stated in Theorem 1, (25) is also verified in simulation.

Next, we consider the case where the output is affected by a noise. Namely, we suppose that the available variables are

$$u(x,t) + \xi(x,t) \quad \forall x \in [Y(t) - \varepsilon, Y(t) + \varepsilon],$$
$$u_x(Y(t),t) + \xi_x(Y(t),t), \quad u_{xxx}(Y(t),t) + \xi_{xxx}(Y(t),t),$$

for a.a. $t > 0$, where $(x, t) \mapsto \xi(x, t) := 0.01\mathcal{N}(x, t)$, with $\mathcal{N}$ a Gaussian white noise of 1dB generated using the MATLAB *wgn* function. We plot in Figure 2 (middle) the evolution of the L^2 norm of the observation error. Finally, we consider the case where there is an uncertainty in the value of λ. Namely, while $\lambda = 4$ in the original KS equation (1), we let $\lambda = 1$ in the observer's dynamics (6). The evolution of the L^2 norm of the observation error is plotted in Figure 2 (right). Although the upperbound in (25) is not verified in the presence of measurement noise and when λ is unknown, which does not contradict the theory derived for $\xi = 0$ and $\lambda = 4$ in both (1) and (6), the L^2 norm of the observation error remains bounded in simulation.

VI. Conclusion

We proposed a novel framework for state estimation using scanning outputs. This framework is illustrated in the context of the nonlinear KS equation in its most general forms, with free boundary conditions and not necessarily-bounded solutions. Although we do not know yet whether using only boundary outputs is enough to design a converging state observer, the flexibility of the proposed sensing framework allowed us to address the estimation problem. For future works, we would like to allow the coefficient λ and the upperbound $\bar{f}$ to be unknown, and to extend our observer-design approach to the two-dimensional KS equation.

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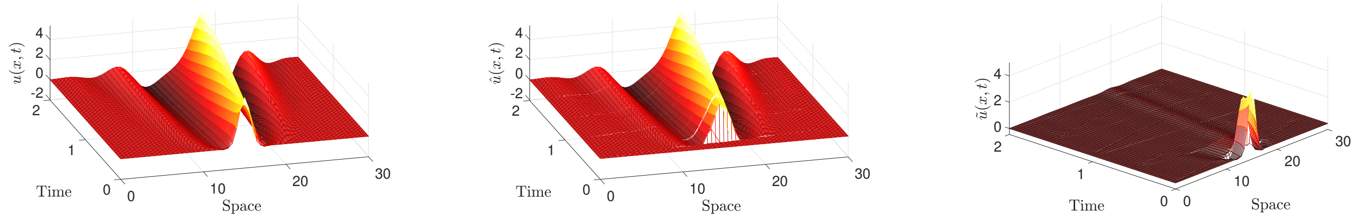

Fig. 1. Solution of the KS equation (1)-(2) (left), the proposed observer (middle), and the resulting observation error $\tilde{u} := u - \hat{u}$ (right).

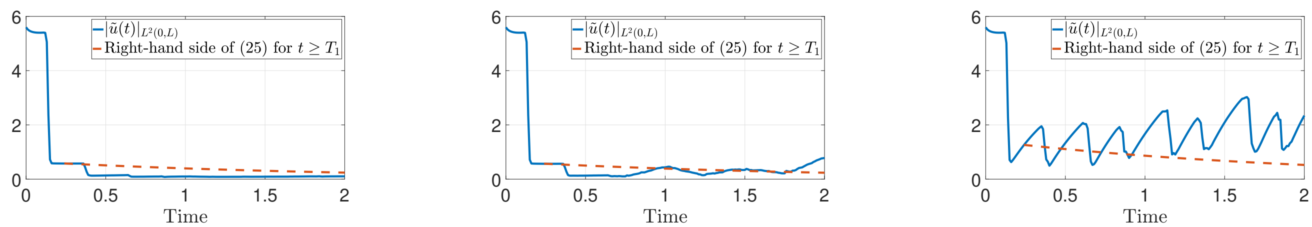

Fig. 2. The L^2 norm of the observation error (blue) versus the right-hand side of the inequality in (25) for $t \geq T_1 := 0.25$ (red). The nominal case in the left. With measurement noise in the middle. With uncertainty in λ in the right.

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Appendix

A. Proof of Proposition 1

To verify (23), we show that

$$\vartheta_1(t) < +\infty \quad \text{and} \quad \vartheta_2(t) < +\infty \quad \text{for a.a. } t > 0, \tag{49}$$
$$\bar{V}_1(t) < +\infty \quad \text{and} \quad \bar{V}_2(t) < +\infty \quad \text{for a.a. } t > 0. \tag{50}$$

We verify the first inequality in (49) by showing that

$$\operatorname*{ess\,sup}_{x \in [0, Y(t)]} |\hat{w}(x,t)| < +\infty \quad \text{for a.a. } t > 0. \tag{51}$$

According to D1 in Definition 2.1,

$$\hat{w}(\cdot,t) \in H^4(0, Y(t)) \quad \text{for a.a. } t > 0 \text{ such that } Y(t) \neq 0.$$

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This implies that

$$\operatorname*{ess\,sup}_{x\in[0,Y(t)]} |\hat{w}(x,t)| < +\infty \quad \text{for a.a. } t>0 \text{ such that } Y(t)\neq 0.$$

Furthermore, having $\operatorname{ess\,sup}_{x=0} |\hat{w}(x,t)| = 0$, (51) follows. The same reasoning applies to show the second inequality in (49).

Now, to verify (50), according to Carathéodory's existence theorem, we need to verify that ϑ_1 and ϑ_2 are locally integrable, i.e., for any bounded interval $I \subset [0,+\infty)$,

$$\int_I \vartheta_1(s)ds < +\infty \quad \text{and} \quad \int_I \vartheta_2(s)ds < +\infty. \tag{52}$$

To show that the first inequality in (52) holds, it is enough to show that

$$\int_I \operatorname*{ess\,sup}_{x\in[0,Y(s)]} |\hat{w}(x,s)|ds < +\infty.$$

We start by introducing the bounded set $\mathcal{I} := I\backslash\{t \in I \,:\, Y(t)=0\}$, and note that

$$\int_I \operatorname*{ess\,sup}_{x\in[0,Y(s)]} |\hat{w}(x,s)|ds = \int_{\mathcal{I}} \operatorname*{ess\,sup}_{x\in[0,Y(s)]} |\hat{w}(x,s)|ds.$$

Furthermore, given $t\in\mathcal{I}$, we have, according to Agmon's inequality (see [13, Lemma A.2]),

$$\begin{aligned}\operatorname*{ess\,sup}_{x\in[0,Y(t)]} |\hat{w}(x,t)|^2 &\leq \hat{w}(0)^2 + 2|\hat{w}(t)|_{L^2(0,Y(t))}|\hat{w}_x(t)|_{L^2(0,Y(t))} \\ &\leq u(0)^2 + |\hat{w}(t)|^2_{L^2(0,Y(t))} + |\hat{w}_x(t)|^2_{L^2(0,Y(t))},\end{aligned}$$

which implies that

$$\begin{aligned}\int_I \operatorname*{ess\,sup}_{x\in[0,Y(s)]} |\hat{w}(x,s)|^2 ds \leq{}& \int_{\mathcal{I}} u(0,s)^2 ds \\ &+ \int_{\mathcal{I}} |\hat{w}(s)|^2_{L^2(0,Y(s))}ds + \int_{\mathcal{I}} |\hat{w}_x(s)|^2_{L^2(0,Y(s))}ds.\end{aligned}$$

Since $\mathcal{I}$ is bounded and $Y(t)\neq 0$ for all $t\in\mathcal{I}$, then, according to the items D3 and D4 in Definition 2.1, we have

$$\max\left\{ \int_{\mathcal{I}} |\hat{w}(s)|^2_{L^2(0,Y(s))}ds, \int_{\mathcal{I}} |\hat{w}_x(s)|^2_{L^2(0,Y(s))}ds \right\} < +\infty.$$

Now, since $u \in L^2_{loc}(0,+\infty; H^4(0,L))$, then $\int_{\mathcal{I}} u(0,s)^2 ds < +\infty$. The same reasoning applies to prove the second inequality in (52).

Finally, (22)-(23) follow under (21), the fact that $|u(Y)| < +\infty$ and $|u_{xxx}(Y)| < +\infty$ a.e. on $[0,+\infty)$, and using item D1 in Definition 2.1 which ensures that $|\hat{w}_{xxx}(Y)| < +\infty$ a.e. on $[0,+\infty)$ when $Y\neq 0$, and $|\hat{v}_{xxx}(Y)| < +\infty$ a.e. on $[0,+\infty)$ when $Y\neq L$.

B. Proof of Lemma 2

Let $I\subset(0,+\infty)$ be a compact interval of non-zero measure such that $Y(t)\neq 0$ for all $t\in I$. We will show that

$$\dot{V}_1 = \frac{3}{4}\dot{Y}\zeta_3^2 + \frac{3}{2}\int_0^Y \tilde{w}(x)\tilde{w}_t(x)dx \quad \text{a.e. on } I. \tag{53}$$

To do so, we consider the function $(s,t)\mapsto \mathcal{W}(s,t) := \tilde{w}(sY(t),t)$ defined for all $(s,t)\in[0,1]\times I$. Note that we have

$$V_1 = \frac{3Y}{4}\int_0^1 \mathcal{W}(s)^2 ds \quad \text{a.e. on } I,$$

and we would like to show that

$$\dot{V}_1 = \frac{3\dot{Y}}{4}\int_0^1 \mathcal{W}(s)^2 ds + \frac{3Y}{2}\int_0^1 \mathcal{W}(s)\mathcal{W}_t(s)ds \quad \text{a.e. on } I. \tag{54}$$

According to Lemma 1.2 in [22, Page 176], to prove the latter identity, it is sufficient to show that $\mathcal{W}\in L^2(I;L^2(0,1))$ and $\mathcal{W}_t\in L^2(I;L^2(0,1))$.

To show that $\mathcal{W}\in L^2(I;L^2(0,1))$, we need to prove that

$$\int_0^1 \mathcal{W}(s,t)^2 ds < +\infty \quad \text{for a.a. } t\in I \tag{55}$$

$$\text{and} \quad \int_I |\mathcal{W}(t)|^2_{L^2(0,1)}dt < +\infty. \tag{56}$$

To prove (55), we use Young's inequality, to obtain

$$\begin{aligned}\int_0^1 \mathcal{W}(s)^2 ds &= \frac{1}{Y}\int_0^Y \tilde{w}(x)^2 dx \\ &\leq \frac{3}{2Y}\int_0^Y \hat{w}(x)^2 dx + \frac{3}{2Y}\int_0^Y u(x)^2 dx \quad \text{a.e. on } I.\end{aligned}$$

Furthermore, since $Y(t)\neq 0$ for all $t\in I$ and Y is continuous, then $\min_{t\in I} Y(t) > 0$. Consequently, the function $t\mapsto 1/Y(t)$ is bounded on I. As a result, to guarantee (55), it is enough to show that

$$\max\left\{\int_0^Y \hat{w}(x)^2 dx, \int_0^Y u(x)^2 dx\right\} < +\infty \quad \text{a.e. on } I.$$

The latter, however, is verified since $u(t)\in H^4(0,L)$ and $\hat{w}(t)\in H^4(0,Y(t))$ for a.a. $t\in I$.

Next, to prove (56), it is enough to show that

$$\max\left\{\int_I |\hat{w}(t)|^2_{L^2(0,Y(t))}dt, \int_I |u(t)|^2_{L^2(0,Y(t))}dt\right\} < +\infty.$$

The latter is true because $u \in L^2_{loc}(0,+\infty; H^4(0,L))$ and $\int_I |\hat{w}(t)|^2_{L^2(0,Y(t))}dt < +\infty$, according to the third item in Definition 2.1.

Similarly, we show that $\mathcal{W}_t\in L^2(I;L^2(0,1))$ by proving that

$$\int_0^1 \mathcal{W}_t(s,t)^2 ds < +\infty \quad \text{for a.a. } t\in I, \tag{57}$$

$$\text{and} \quad \int_I |\mathcal{W}_t(t)|^2_{L^2(0,1)}dt < +\infty. \tag{58}$$

To do so, we note that

$$\mathcal{W}_t = \dot{Y}s\tilde{w}_x + \tilde{w}_t \quad \text{a.e. on } I.$$

As a result, (57) follows if we show that

$$\max\left\{\int_0^Y \hat{w}_t(x)^2 dx, \int_0^Y u_t(x)^2 dx\right\} < +\infty \text{ and}$$
$$\max\left\{\int_0^Y \hat{w}_x(x)^2 dx, \int_0^Y u_x(x)^2 dx\right\} < +\infty \text{ a.e. on } I.$$

However, the latter inequalities are verified because $u_t\in L^2(0,L)$ a.e. on I, and, according to the second item in Definition 2.1, $\hat{w}_t\in L^2(0,Y)$ a.e. on I. Furthermore, $u_x\in L^2(0,L)$ a.e. on I, and, according to D1 in Definition 2.1, $\hat{w}_x\in L^2(0,Y)$ a.e. on I.

Finally, to prove (58), it is sufficient to show that

$$\max\left\{\int_I |\hat{w}_x(t)|^2_{L^2(0,Y(t))}dt, \int_I |u_x(t)|^2_{L^2(0,Y(t))}dt\right\} < +\infty,$$
$$\max\left\{\int_I |\hat{w}_t(t)|^2_{L^2(0,Y(t))}dt, \int_I |u_t(t)|^2_{L^2(0,Y(t))}dt\right\} < +\infty.$$

The latter inequalities hold because $u\in L^2_{loc}(0,+\infty;H^4(0,L))$, $u_t\in L^2_{loc}(0,+\infty;L^2(0,L))$, and because of the third and fourth items in Definition 2.1.

Now that we have shown that $\mathcal{W}\in L^2(I;L^2(0,1))$ and $\mathcal{W}_t\in L^2(I;L^2(0,1))$, we can invoke Lemma 1.2 in [22, Page 176] to conclude that the identity (54) holds a.e. on I. This inequality can be rewritten as

$$\dot{V}_1 = \frac{3\dot{Y}}{4Y}\int_0^Y \tilde{w}(x)^2 dx + \frac{3}{2}\int_0^Y \tilde{w}(x)\left(\frac{x\dot{Y}}{Y}\tilde{w}_x(x) + \tilde{w}_t(x)\right)dx.$$

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Using integration by parts, we find

$$\int_0^Y x\tilde{w}(x)\tilde{w}_x(x)dx = \frac{Y}{2}\zeta_3^2 - \frac{1}{2}\int_0^Y \tilde{w}(x)^2 dx.$$

Combining the latter two equations, we obtain (53). Now, by setting $\varphi := \tilde{w}(t)$ in (26) for a.a. $t \in I$, we get, a.e. on I,

$$\begin{aligned}\dot{V}_1 &= \frac{3}{4}\dot{Y}\zeta_3^2 + \frac{3}{2}\int_0^Y \tilde{w}(x)\tilde{w}_t(x)dx \\ &= \frac{3}{4}\dot{Y}\zeta_3^2 + \frac{3}{2}\zeta_1\zeta_3 - \frac{3}{2}\int_0^Y \tilde{w}(x)\Big(\tilde{w}(x)\tilde{w}_x(x) + \hat{w}(x)\tilde{w}_x(x) \\ &+ \hat{w}_x(x)\tilde{w}(x) + \lambda\tilde{w}_{xx}(x) + \tilde{w}_{xxxx}(x) - f(x)\Big)dx. \end{aligned} \tag{59}$$

Next, using integration by parts, we obtain

$$\int_0^Y \tilde{w}(x)^2\tilde{w}_x(x)dx = \frac{1}{3}\left[\tilde{w}(x)^3\right]_0^Y = \frac{1}{3}\zeta_3^3. \tag{60}$$

Similarly, we have

$$\int_0^Y \tilde{w}(x)\tilde{w}_{xx}(x)dx = \left[\tilde{w}(x)\tilde{w}_x(x)\right]_0^Y - \int_0^Y \tilde{w}_x(x)^2 dx \tag{61}$$

$$= -\int_0^Y \tilde{w}_x(x)^2 dx. \tag{62}$$

$$\begin{aligned}\int_0^Y \tilde{w}(x)\tilde{w}_{xxxx}(x)dx &= \left[\tilde{w}(x)\tilde{w}_{xxx}(x) - \tilde{w}_x(x)\tilde{w}_{xx}(x)\right]_0^Y \\ &+ \int_0^Y \tilde{w}_{xx}(x)^2 dx \\ &= \tilde{w}_{xxx}(Y)\zeta_3 + \int_0^Y \tilde{w}_{xx}(x)^2 dx. \end{aligned} \tag{63}$$

$$\begin{aligned}\int_0^Y \hat{w}_x(x)\tilde{w}(x)^2 dx &= \left[\hat{w}\tilde{w}^2\right]_0^Y - 2\int_0^Y \hat{w}(x)\tilde{w}(x)\tilde{w}_x(x)dx \\ &= (u(Y) - \zeta_3)\zeta_3^2 - 2\int_0^Y \hat{w}(x)\tilde{w}(x)\tilde{w}_x(x)dx. \end{aligned} \tag{64}$$

By combining (59)-(64), we find

$$\begin{aligned}\dot{V}_1 &= \zeta_3^3 + \frac{3}{4}\big(\dot{Y} - 2u(Y)\big)\zeta_3^2 - \frac{3}{2}\tilde{w}_{xxx}(Y)\zeta_3 \\ &+ \frac{3}{2}\int_0^Y f(x)\tilde{w}(x)dx + \frac{3}{2}\zeta_1\zeta_3 \\ &+ \frac{3}{2}\int_0^Y \hat{w}(x)\tilde{w}(x)\tilde{w}_x(x)dx \\ &+ \frac{3}{2}\Big(\lambda\int_0^Y \tilde{w}_x(x)^2 dx - \int_0^Y \tilde{w}_{xx}(x)^2 dx\Big). \end{aligned} \tag{65}$$

Next, using Young's inequality, we conclude that

$$\int_0^Y \hat{w}(x)\tilde{w}(x)\tilde{w}_x(x)dx \le \frac{1}{2}\int_0^Y \big(\hat{w}(x)^2\tilde{w}(x)^2 + \tilde{w}_x(x)^2\big)dx,$$

which implies that

$$\begin{aligned}\int_0^Y \hat{w}(x)\tilde{w}(x)\tilde{w}_x(x)dx &\le \frac{1}{2}\operatorname*{ess\,sup}_{x\in(0,Y)} |\hat{w}(x)|^2 \int_0^Y \tilde{w}(x)^2 dx \\ &+ \frac{1}{2}\int_0^Y \tilde{w}_x(x)^2 dx \\ &\le \frac{2}{3}\operatorname*{ess\,sup}_{x\in(0,Y)} |\hat{w}(x)|^2 V_1 + \frac{1}{2}\int_0^Y \tilde{w}_x(x)^2 dx. \end{aligned} \tag{66}$$

Moreover, applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\int_0^Y f(x)\tilde{w}(x)dx &\le \sqrt{\int_0^Y f(x)^2 dx \int_0^Y \tilde{w}(x)^2 dx} \\ &\le \frac{2\sqrt{3}}{3}\left(\sqrt{\int_0^Y f(x)^2 dx}\right)\sqrt{V_1}. \end{aligned} \tag{67}$$

Next, given $\sigma > 0$ and using Young's inequality, we obtain

$$\int_0^Y f(x)\tilde{w}(x)dx \le \frac{\sqrt{3}\sigma}{3}V_1 + \frac{\sqrt{3}}{3\sigma}\int_0^Y f(x)^2 dx.$$

Combining (65) and (66)-(67), we find

$$\begin{aligned}\dot{V}_1 &\le \zeta_3^3 + \frac{3}{4}\big(\dot{Y} - 2u(Y)\big)\zeta_3^2 - \frac{3}{2}\tilde{w}_{xxx}(Y)\zeta_3 + \frac{3}{2}\zeta_1\zeta_2 \\ &+ \left(\operatorname*{ess\,sup}_{x\in(0,Y)} |\hat{w}(x)|^2 + \frac{\sigma\sqrt{3}}{2}\right)V_1 + \frac{\sqrt{3}}{2\sigma}\int_0^Y f(x)^2 dx \\ &+ \frac{3}{2}\left(\left(\lambda + \frac{1}{2}\right)\int_0^Y \tilde{w}_x(x)^2 dx - \int_0^Y \tilde{w}_{xx}(x)^2 dx\right). \end{aligned} \tag{68}$$

Now, to construct an upperbound on the term

$$\left(\lambda + \frac{1}{2}\right)\int_0^Y \tilde{w}_x(x)^2 dx - \int_0^Y \tilde{w}_{xx}(x)^2 dx,$$

we employ Wirtinger's inequality (see Section 7.7 in [9]) and the inequality 23.1 in [19, Page 84]. Having

$$(\lambda + 1/2)\,(Y/\pi)^2 \le 1 \tag{69}$$

and since $\tilde{w}_x(0) = \tilde{w}_x(Y) = 0$, then according to Wirtinger's inequality, we have

$$\int_0^Y \tilde{w}_x(x)^2 dx \le \left(\frac{Y}{\pi}\right)^2 \int_0^Y \tilde{w}_{xx}(x)^2 dx,$$

which implies that

$$\begin{aligned}&\left(\lambda + \frac{1}{2}\right)\int_0^Y \tilde{w}_x(x)^2 dx - \int_0^Y \tilde{w}_{xx}(x)^2 dx \\ &\le \left(\left(\lambda + \frac{1}{2}\right)\left(\frac{Y}{\pi}\right)^2 - 1\right)\int_0^Y \tilde{w}_{xx}(x)^2 dx. \end{aligned} \tag{70}$$

Consequently, using (69) and (70), we obtain

$$\left(\lambda + \frac{1}{2}\right)\int_0^Y \tilde{w}_x(x)^2 dx - \int_0^Y \tilde{w}_{xx}(x)^2 dx \le 0. \tag{71}$$

Now, we suppose that $Y > \pi/\sqrt{\lambda + 1/2}$. Using the inequality 23.1 in [19, Page 84], we obtain

$$\begin{aligned}\int_0^Y \tilde{w}_x(x)^2 dx &\le \left(\lambda + \frac{1}{2} + \frac{12}{Y^2}\right)\int_0^Y \tilde{w}(x)^2 dx \\ &+ \frac{1}{\lambda + \frac{1}{2}}\int_0^Y \tilde{w}_{xx}(x)^2 dx. \end{aligned} \tag{72}$$

Multiplying both sides of (72) by $(\lambda + 1/2)$, we find

$$\begin{aligned}&\left(\lambda + \frac{1}{2}\right)\int_0^Y \tilde{w}_x(x)^2 dx - \int_0^Y \tilde{w}_{xx}(x)^2 dx \\ &\le \frac{4}{3}\left(\lambda + \frac{1}{2}\right)\left(\lambda + \frac{1}{2} + \frac{12}{Y^2}\right)V_1. \end{aligned} \tag{73}$$

Combining (71) and (73), we obtain

$$\frac{3}{2}\left(\left(\lambda + \frac{1}{2}\right)\int_0^Y \tilde{w}_x(x)^2 dx - \int_0^Y \tilde{w}_{xx}(x)^2 dx\right) \le \theta_1 V_1, \tag{74}$$

where θ_1 is as defined in (17). Finally, from (68) and (74), (37) follows. We prove (38) in the exact same way.