Adaptive Space-Time Distributed Parameter and Input Estimation in Heat Transport with Unknown Bounds[1]

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Abstract—: In this paper, we discuss on-line adaptive estimation of distributed diffusion and source term coefficients for a non-homogeneous linear parabolic partial differential equation describing heat transport. An estimator is defined in the infinite-dimensional framework having the system state and the parameters' estimate as its states. Our scheme allows to estimate spatially distributed and space-time distributed parameters. While the parameters convergence depends on the plant signal richness assumption, the state convergence is established using the Lyapunov approach. Since the estimator is infinite-dimensional, the b-splines Galerkin finite element method is used to implement it. In silico simulations are provided to illustrate the performance of the proposed approach.

I. Introduction

Distributed parameter systems (DPS) widely exist in industrial processes: thermal processes, fluid processes and transport-reaction processes. These physical and chemical systems are governed by partial differential equations (PDE) and complex spatiotemporal nonlinear dynamics. In many situations, it is difficult to get an accurate nominal PDE description due to incomplete physical or chemical knowledge (unknown system parameters, unknown disturbances...). These uncertainties make the modeling problem tedious. Three different problems in DPS are of prime interest: (i) model reduction (ii) system identification ("black-box modeling") and (iii) parameter estimation ("grey-box modeling"), where the PDE structure is available and only some parameters need to be known. The latter, is the one of our interest.

Heat transport is a complex DPS phenomenon where various mechanism of heat exchange occur. It is described by a general non-homogeneous second order linear parabolic partial differential equation (PDE). In this work, we consider one-dimensional heat transport governed by a diffusion-reaction PDE with mixed Dirichlet-Neumann boundary conditions and where the source term (the input power adsorbed by the process) is distributed and poorly known. In our previous works ([1], [2]), we attempted to solve this problem for thermonuclear heat transport in an early lumping estimation approach. First the problem was discretized using the Galerkin formulation and then a modified Kalman filter was applied to estimate both the diffusion coefficient and the unknown input. In order to investigate the problem of heat transport in its original description (a DPS process) late

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lumping estimation approach is analysed. In this paper, we focus on the infinite-dimensional framework using on-line or adaptive estimation techniques.

Off-line methods for functional parameters in DPS were extensively studied both in finite and infinite-dimensional frameworks ([3], [4], etc.). The identification methods were based mainly on output least squares and maximum likelihood estimators. These techniques lead generally to nonlinear optimization problems. An on-line estimator utilizes the PDE characteristics and consequently the inverse problem remains linear.

Adaptive estimation for infinite dimensional dynamic systems with both constant, spatially varying and time-varying parameters with known input was first addressed in [5] and their earlier works (for example: Dr. Scondo and Dr. Demetriou's Ph. D theses). They also established the abstract framework of this problem using functional analysis tools and Sobolev spaces properties. In [6], constructively enforceable identifiability conditions based on manipulable quantities were introduced for the first time. Unlike [5], the proposed adaptive estimator used the PDE features to reduce the order of the spatial derivatives by obviating the repetition of the spatial derivative structure of the plant in the tuning laws. In [7], model reference adaptive control (MRAC) of a linear parabolic partial differential equation with time-varying coefficients was treated.

Two main contributions are presented in this work. First, the identifiability conditions of the simultaneous diffusion/input estimation problem are shown. Since the input is unknown, these conditions are passive. Then, spatially varying and space-time input/parameters' simultaneous on-line estimation is considered. The problem of spatially varying parameters can be viewed as a particular case of space-time DPS systems. In this paper, it serves both as an introduction to our inverse problem, where consistent results can be found and as an extension of [6]'s work to the joint input/parameter identification problem In the second part, distributed space-time varying input/parameter are examined. Inspired by [7] paper, we demonstrate that a region of convergence of the state error can be expressed explicitly as a function of the identifiers' tuning parameters. Throughout this paper, we assume that distributed sensing and measurements are available. The question of input estimation is not only related to heat transport but arises, to cite few, in fault detection, machine tool and manipulator applications, chaotic systems and general inverse problems.

This paper is organized as follows. The heat transport

model and the framework of our PDE problem are presented in Section II. In Section III, we treat the diffusion and source term identifiability conditions. The adaptive estimators for functional state, input and parameters are considered in Section IV. In order to illustrate the performance of the proposed identifiers, computer simulations are carried out in Section V.

II. Heat transport model

In general heat transfer textbooks, the heat equation is derived using conservation of energy principle and Fourier's law of heat conduction [8]. Its one dimensional linear form is given by

$$
\begin{cases}
\dfrac{\partial T}{\partial t} = \dfrac{1}{x}\dfrac{\partial}{\partial x}\left(x\,\chi_e(x,t)\,\dfrac{\partial T}{\partial x}\right) - \dfrac{1}{\tau}T(x,t) + S(x,t); \\[2mm]
\dfrac{\partial T}{\partial x}(0,t) = 0\ ;\ T(1,t) = 0; \\[2mm]
T(x,0) = 0;\quad x\in\Omega;\quad t\in]0,t_f],
\end{cases}
\tag{1}
$$

where t is the time, x is the normalized spatial variable i.e. $x \in \Omega =]0,1[$, χ_e is the diffusion coefficient, $\tau (< \infty)$ is a damping time modelling energy losses, T is the temperature S is the power density absorbed from an external heating system.
In system (1), the second and third equations represent initial and boundary conditions, chosen to guaranty the symmetry and boundedness of the solution near zero. The value $T(1,t) = T(x,0) = 0$ refers to an equilibrium temperature level and not to the absolute zero temperature. The diffusion coefficient χ_e is assumed to belong to the space $Q_\chi = \{f : f(x,t) \in C^0(0,t_f;C^1(\Omega)), \exists c_1, c_2 \in \mathbb{R}_*^+\ c_2 > f(x,t) > c_1 > 0\}$, the reaction coefficient τ in $\mathbb{R}_*^+$ and the source term S in $Q_S := C^0(0,t_f\,;\,C^0(\Omega)) \cap L^2(0,t_f,L^2(\Omega))$. These assumptions guarantee the existence, uniqueness and differentiability of the classical solution of PDE (1) in $Q_T := C^1(0,t_f;C^2(\Omega))$(see [9], chapter 07, page 375).
In the following, τ is considered as a known variable and we aim to estimate χ_e and S. The raison for which τ is assumed to be a constant coefficient is related to the complexity of proving the structural identifiability of the three parameters.

III. Diffusion and source term identifiability

The input/parameter estimation convergence is necessary guaranteed if we can ensure the problem identifiability: whether it is possible to uniquely extract the solution for the model unknown variables from the measurements.
In the finite-dimensional framework, this problem has been extensively studied (see [10] and references therein) both for structural (noise-free model) and practical aspects. In adaptive estimation techniques, this is equivalent to ensure plant signals richness. The finite-dimensional notion of persistence of excitation was extended to infinite-dimensional systems first by [5] and [11]. Later, in [6], constructively enforceable identifiability conditions were given and the construction of persistently exciting input was shown. In this paper, the structural identifiability question is treated in the spirit of [6] paper, extended to deal with both diffusion/source term reconstruction and the only available input under which identifiability is considered are the PDE's (1) boundary conditions.

Definition: ([6]) A set of parameters $\{\chi_e(x,t),\ S(x,t)\}$ of the PDE (1) is said to be identifiable with the corresponding boundary conditions if and only if:

$$
\forall x \in]0,1[, \forall t \geq 0,\ \forall \overline{\chi}_e \neq \chi_e,\ \forall\ \overline{S} \neq S :
$$

$$
div\left(\Delta\chi_e\frac{\partial T}{\partial x}\right) + \Delta S = 0 \Rightarrow \Delta\chi_e = \Delta S = 0,
\tag{2}
$$

where div is the divergence operator in the cylindrical coordinates supposing a gradient in the x direction only:
$$
div(.) := \frac{1}{x}\frac{\partial}{\partial x}(x\ .),\ \Delta\chi_e = \chi_e - \overline{\chi}_e\ \text{and}\ \Delta S = S - \overline{S}.
$$

The concept of persistent excitation relies on the ability of finding the Fourier expansion of the plant state on an arbitrary orthonormal basis in $L_2(0,1)$. $T(x,t)$ can be written as:

$$
T(x,t) = \sum_{n=0}^{\infty} l_n(t)\cos(\pi n x).
\tag{3}
$$

where the Fourier coefficients l_n are linearly independent functions (for more details see [6]). Sufficient conditions for simultaneous diffusion and source term identification are given as follows.
Theorem 3.1: If the boundary conditions (and the input) of the PDE (1) generate a persistent excitation of the system (such that the Fourier coefficients of the plant state are linearly independent), then the parameter χ_e and the external input S are identifiable (with these boundary conditions).
Proof 3.1: Substituting the Fourier expansion of T given in (3) into (2) yields:

$$
\sum_{n=0}^{\infty}\ \{[div(\Delta\chi_e)n\pi l_n(t) + \frac{n\pi}{x}\Delta\chi_e l_n(t)]sin(n\pi x) +
$$
$$
+ (n\pi)^2\Delta\chi_e l_n(t)cos(n\pi x)\} - \Delta S = 0.
\tag{4}
$$

The input S can also be written as a Fourier series:

$$
S(x,t) = \sum_{n=0}^{\infty}\omega_n(t)cos(n\pi x)
$$

where $\{\omega_n(t)\}_{n=0}^{n=\infty}$ are linearly independent. Substituting S in (4) gives:

$$
\sum_{n=0}^{\infty}\{[div(\Delta\chi_e)n\pi l_n(t) + \frac{n\pi}{x}\Delta\chi_e l_n(t)]sin(n\pi x) +
$$
$$
+ [(n\pi)^2\Delta\chi_e l_n(t) - \Delta\omega_n(t)]cos(n\pi x)\} = 0.
\tag{5}
$$

Since the sets $\{cos(n\pi x)\}$ and $\{sin(n\pi x)\}$ have no intersecting zero and $\{l_n(t)\}_{n=0}^{\infty}$ are linearly independent (persistent excitations hypothesis), we conclude that (5) implies:

$$
\Delta\chi_e = 0 \text{ and } (\Delta\omega_n = 0 \Leftrightarrow \Delta S = 0).
$$

Remark: Since the input is unknown and the aim of this work is to estimate it simultaneously with the diffusion coefficient, unlike what was done in [6], input persistent generators cannot be constructed. The persistent excitation of the plant is investigated *a posteriori*, once the measurements are available by checking the linear dependence of the plant Fourier coefficients unless a simulator on which tests can be carried out is accessible.

IV. ADAPTIVE ESTIMATOR DESIGN FOR DISTRIBUTED TIME-SPACE INPUT AND DIFFUSION COEFFICIENT

For sufficiently rich signals, the simultaneous estimation of the input and the diffusion coefficient can be achieved using an adaptive estimator. The adaptive law is developed based on stability considerations or using simple optimization techniques to minimize the output error equation.

In this section, we first assume that the parameters (S and χ_e) are time independent. Our initial objective is to demonstrate that under this assumption, a stable adaptive estimator for the simultaneous problem exists and guarantees the L_2 and point-wise convergence of both state and parameters (input and diffusion coefficient). The problem of distributed slowly time varying coefficients and time-varying parameters with known upper bounds will be addressed in the next subsection.

A. Adaptive estimator for spatially varying parameters and input

The adaptive identifier is a model-based estimator. It takes the form of PDE (1) to which an innovation term (correction) is added and a gradient-type update law for the parameters' estimate is associated. Supposing that distributed sensors are available and measure the system state, this estimator is described by:

$$
\begin{cases}
\dfrac{\partial \hat{T}}{\partial t} = \dfrac{1}{x}\dfrac{\partial}{\partial x}\left(x\,\hat{\chi}_e(x,t)\dfrac{\partial \hat{T}}{\partial x}\right) - \dfrac{1}{\tau}\hat{T}(x,t) + \hat{S}(x,t) \\
\qquad -\vartheta_0\left(\hat{T}(x,t) - T(x,t)\right); \\[2mm]
\dfrac{\partial \hat{T}}{\partial x}(0,t) = \hat{T}(1,t) = 0; \quad \hat{T}(x,0) = \hat{T}_0(x) \geq 0;
\end{cases}
\tag{6}
$$

$$
\begin{cases}
\dfrac{\partial \hat{\chi}_e}{\partial t} = \vartheta_1\dfrac{\partial}{\partial x}\left(\hat{T} - T\right)\dfrac{\partial \hat{T}}{\partial x}; \hat{\chi}_e(x,0) = \hat{\chi}_{e_0}(x); \\[2mm]
\dfrac{\partial \hat{S}}{\partial t} = -\vartheta_2\left(\hat{T} - T\right); \hat{S}(x,0) = \hat{S}_0(x); \\[2mm]
\forall\, x \in\,]0,1[\,;\ t \geq 0,
\end{cases}
\tag{7}
$$

where $\vartheta_i \geq 0$, $i = 0,1,2$ are the adaptation gains, $\hat{\chi}_{e_0}(x) > 0$ is a smooth function and $S_0(x)$ is a continuous function.

Theorem 4.1: If the plant (1) is identifiable (under persistent excitations), the adaptive identification law given by (6) combined with the parameters identifiers in (7) ensure the L_2 convergence of the state and parameters deviations.

Proof 4.1: Let us first define the state and parameters deviations

$$
\begin{cases}
\Delta T = \hat{T} - T, \dfrac{\partial \Delta T}{\partial x}(0,t) = \Delta T(1,t) = 0; \\
\Delta\chi_e(x,t) = \hat{\chi}_e(x,t) - \chi_e(x); \\
\Delta S(x,t) = \hat{S}(x,t) - S(x);
\end{cases}
\tag{8}
$$

and their derivatives:

$$
\begin{cases}
\dfrac{\partial \Delta T}{\partial t} = div\left(\chi_e(x)\dfrac{\partial \Delta T}{\partial x}\right) + div\left(\Delta\chi_e(x,t)\dfrac{\partial \hat{T}}{\partial x}\right) \\
\qquad -(\dfrac{1}{\tau} + \vartheta_0)\Delta T(x,t) + \Delta S. \\[2mm]
\dfrac{\partial \Delta\chi_e}{\partial t} = \vartheta_1\dfrac{\partial \hat{T}}{\partial x}\dfrac{\partial \Delta T}{\partial x}. \\[2mm]
\dfrac{\partial \Delta S}{\partial t} = -\vartheta_2\ \Delta T(x,t).
\end{cases}
\tag{9}
$$

Since $\hat{\chi}_{e_0}(x) > 0$ and $\tau \in \mathbb{R}_*^+$ are bounded, $\hat{S}(x)$ is a continuous bounded function. There exists a unique local solution for the global system (1), (6) and (7). Thus the problem is well-posed and we introduce the following Lyapunov functional

$$
V(t) = \frac{1}{2}\int_0^1 \left([\Delta T(x,t)]^2 + \frac{1}{\vartheta_1}[\Delta\chi_e(x,t)]^2 \right.
$$
$$
\left. + \frac{1}{\vartheta_2}[\Delta S(x,t)]^2\right)dx.
\tag{10}
$$

Taking into account the system (9) and using the Gauss' divergence formula:

$$
\int_0^1 div\left(\chi_e\frac{\partial \Delta T}{\partial x}\,\Delta T\right)dx = \chi_e\frac{\partial \Delta T}{\partial x}\Delta T\Big|_0^1
$$

and:

$$
div\left(\chi_e\frac{\partial \Delta T}{\partial x}\,\Delta T\right) = \Delta T\, div\left(\chi_e\frac{\partial \Delta T}{\partial x}\right) + \chi_e\left(\frac{\partial \Delta T}{\partial x}\right)^2
$$

we obtain the following integration by parts for the divergence term:

$$
\int_0^1 div\left(\chi_e\frac{\partial \Delta T}{\partial x}\right)\Delta T\,dx = \chi_e\frac{\partial \Delta T}{\partial x}\Delta T\Big|_0^1 - \int_0^1\chi_e\left(\frac{\partial \Delta T}{\partial x}\right)^2 dx.
$$

The time derivative of the Lyapunov functional (10) is given

by:

$$\dot{V}(t) = \chi_e(x,t)\frac{\partial \Delta T}{\partial x}\Delta T(x,t)\Big|_0^1 - \int_0^1 \chi_e(x,t)\left(\frac{\partial \Delta T}{\partial x}\right)^2 dx$$

$$+\Delta\chi_e(x,t)\frac{\partial \hat{T}}{\partial x}\Delta T(x,t)\Big|_0^1 - \int_0^1 \Delta\chi_e(x,t)\frac{\partial \Delta T}{\partial x}\frac{\partial \hat{T}}{\partial x}dx$$

$$-\int_0^1 \left(\frac{1}{\tau}+\vartheta_0\right)[\Delta T(x,t)]^2 dx + \int_0^1 \Delta S(x,t)\Delta T(x,t)dx$$

$$+\int_0^1 \Delta\chi_e(x,t)\frac{\partial \hat{T}}{\partial x}\frac{\partial \Delta T}{\partial x}dx - \int_0^1 \Delta S(x,t)\Delta T(x,t)dx$$

$$= -\int_0^1 \chi_e(x,t)\left(\frac{\partial \Delta T}{\partial x}\right)^2 dx - \int_0^1 (\frac{1}{\tau}+\vartheta_0)[\Delta T(x,t)]^2 dx$$

$$\leq -\int_0^1 (\frac{1}{\tau}+\vartheta_0)[\Delta T(x,t)]^2 dx \leq 0$$

This proves the boundedness of Lyapunov functional (10) for all $t \geq 0$ and L_2 boundedness of system solutions (6),(7). In this special case, the invariance principle can be used (see [6] and references therein). Therefore, the trajectories of system (6),(7) converge to the maximal invariant subset of a set of solutions of (6),(7), for which $\dot{V} = 0$. This implies $\Delta T = 0$ and leads to the following expression:

$$div(\Delta\chi_e(x,t)\frac{\partial T}{\partial x}) + \Delta S(x,t) = 0; \ \forall x \in]0,1[, t \geq 0 \quad (12)$$

With the identifiability hypothesis, it follows that:

$$\Delta\chi_e(x,t) = \Delta S(x,t) = 0; \ \forall x \in]0,1[, t \geq 0.$$

and thus, we deduce that

$$\lim_{t\to+\infty}\int_0^1 \left\{(\Delta T)^2 + (\Delta S)^2 + (\Delta\chi_e)^2\right\} dx = 0.$$

Finally, using the same methodology as the one proposed in [6], parameters' point-wise convergence can be asserted.

B. Adaptive estimation of space-time parameter and input with unknown bounds

In this subsection, we discuss space-time varying $\chi_e(x,t)$ and $S(x,t)$. Considering the same Lyapunov-like function (10), its derivative in (11) becomes:

$$\dot{V}(t) = \int_0^1 \Delta T\frac{\partial \Delta T}{\partial t}dx + \frac{1}{\vartheta_1}\int_0^1 \Delta\chi_e\frac{\partial \hat{\chi}_e}{\partial t}dx$$

$$+\frac{1}{\vartheta_2}\int_0^1 \Delta S\frac{\partial \hat{S}}{\partial t}dx + \frac{1}{\vartheta_1}\int_0^1 \Delta\chi_e\frac{\partial \chi_e}{\partial t}dx \quad (13)$$

$$+\frac{1}{\vartheta_2}\int_0^1 \Delta S\frac{\partial S}{\partial t}dx$$

Retaining the same adaptation laws (6),(7) in (13), yields to:

$$\dot{V}(t) = -\int_0^1 \chi_e(x,t)\left(\frac{\partial \Delta T}{\partial x}\right)^2 dx$$

$$-\int_0^1 (\frac{1}{\tau}+\vartheta_0)[\Delta T(x,t)]^2 dx$$

$$-\frac{1}{\vartheta_1}\int_0^1 \Delta\chi_e\frac{\partial \chi_e}{\partial t}dx - \frac{1}{\vartheta_2}\int_0^1 \Delta S\frac{\partial S}{\partial t}dx \quad (14)$$

$$\leq -(\frac{1}{\tau}+\vartheta_0)\int_0^1 [\Delta T(x,t)]^2 dx$$

$$-\frac{1}{\vartheta_1}\int_0^1 \Delta\chi_e\frac{\partial \chi_e}{\partial t}dx - \frac{1}{\vartheta_2}\int_0^1 \Delta S\frac{\partial S}{\partial t}dx$$

Assuming that Q_χ and Q_S are compact Banach spaces such that: $\exists c_3 \in \mathbb{R}_*^+ : \|S(x,t)\|_2 \leq c_3$, and $\|\dot{\chi}_e(x,.)\|_2 \in L_1(0,\infty)$, gives

$$\int_0^\infty \left(\frac{1}{\tau}+\vartheta_0\right)\|\Delta T(x,.)\|_2 dt \leq V(0) - V(\infty)$$

$$+\frac{c_2}{\vartheta_1}\int_0^\infty \|\dot{\chi}_e\|_2 dt + \frac{c_3}{\vartheta_2}\int_0^\infty \|\dot{S}\|_2 dt < \infty \quad (15)$$

Therefore, $\lim_{t\to+\infty}\|\Delta T\|_2(.,t) = 0$.

The convergence of the state estimation error to zero is due principally to L^∞-time-boundedness assumptions on $\dot{\chi}_e$ and $\dot{S}$. More general problem would be one on which parameters χ_e, S, $\dot{\chi}_e$ and $\dot{S}$ vary in unknown fashion but only L^2-bounded. In this case, L^2 convergence of the state error can no long hold and adaptation laws (7) have to be adjusted in order to take into account parameter time-variations.

To this end, let us consider the following modified adaptation law: $\forall x \in]0,1[; t \geq 0$,

$$\begin{cases} \dfrac{\partial \hat{\chi}_e}{\partial t} = -\sigma_e\,\hat{\chi}_e(x,t) + \vartheta_1\left(\dfrac{\partial \Delta T}{\partial x}\dfrac{\partial \hat{T}}{\partial x} + g_e(x,t)\right), \\[2mm] \hat{\chi}_e(x,0) = \hat{\chi}_{e_0}(x); \\[2mm] \dfrac{\partial \hat{S}}{\partial t} = -\sigma_s\,\hat{S}(x,t) - \vartheta_2\left(\Delta T + g_s(x,t)\right), \\[2mm] \hat{S}(x,0) = \hat{S}_0(x), \end{cases} \quad (16)$$

where functions g_e and g_s are given by:

$$\begin{cases} g_e(x,t) = -\dfrac{\mu_e^2}{\mu_e\,\|\hat{\chi}_e(x,.)\|_2 + \epsilon_e}\,\hat{\chi}_e(x,t), \\[2mm] \mu_e \geq \|f_e(x,t)\|_2, \ f_e(x,t) \triangleq -\lambda_e\left(\chi_e(x,t) + \dfrac{1}{\sigma_e}\dot{\chi}_e(x,t)\right), \\[2mm] g_s(x,t) = \dfrac{\mu_s^2}{\mu_s\,\|\hat{S}(x,.)\|_2 + \epsilon_s}\,\hat{S}(x,t), \\[2mm] \mu_s \geq \|f_s(x,t)\|_2, \ f_s(x,t) \triangleq -\lambda_s\left(S(x,t) + \dfrac{1}{\sigma_s}\dot{S}_e(x,t)\right), \\[2mm] \lambda_e = \dfrac{\sigma_e}{\vartheta_1}, \lambda_s = \dfrac{\sigma_s}{\vartheta_2}, \epsilon_e > 0, \ \sigma_e > 0, \ \epsilon_s > 0, \sigma_s > 0. \end{cases} \quad (17)$$

The form in which parameter time-variations are considered in adaptation laws (16) is deduced from [7]. In that work, the addressed problem was the Model Reference Adaptive Control (MRAC) of a parameter time-varying parabolic partial differential equation. In our work, a conversion and extension to the problem of simultaneous input and parameter estimation is examined.

Note that, the terms $-\sigma_e\,\hat{\chi}_e(x,t)$ and $-\sigma_s\,\hat{S}(x,t)$ are the $\sigma-modification\ terms$, added to enhance parameter convergence and to make the adaptation laws more robust with respect to bounded unknown model dynamics [12]. However, these addad terms may deteriorate some convergence properties and the asymptotic convergence of the state error.

Theorem 4.2: Consider the state parameter and input estimators in (6),(16) and (17). If the model is identifiable (persistently excited), then ΔT, $\Delta \chi_e$ and ΔS are uniformly ultimately L^2-bounded. The uniform ultimate boundedness region can be made arbitrary small by suitable choice of σ_e, σ_s, ϵ_e, ϵ_s, ϑ_1, ϑ_2.

Proof 4.2: Substituting (16) in (13) gives

$$\dot{V} = -\left(\frac{1}{\tau} + \vartheta_0\right) \int_0^1 (\Delta T)^2 dx - \int_0^1 \chi_e \left(\frac{\partial \Delta T}{\partial t}\right)^2 dx$$

$$+ \int_0^1 \Delta\chi_e g_e dx - \frac{\sigma_e}{\vartheta_1} \int_0^1 \Delta\chi_e \hat{\chi}_e dx$$

$$- \frac{1}{\vartheta_1} \int_0^1 \Delta\chi_e \dot{\chi}_e dx - \int_0^1 \Delta S g_s dx - \frac{\sigma_s}{\vartheta_2} \int_0^1 \Delta S \hat{S} dx$$

$$- \frac{1}{\vartheta_2} \int_0^1 \Delta S \dot{S} dx$$

Let us note

$$V_0 = \left(\frac{1}{\tau} + \vartheta_0\right) \int_0^1 (\Delta T)^2 dx - \int_0^1 \chi_e \left(\frac{\partial \Delta T}{\partial t}\right)^2 dx,\ V_0 \geq 0,$$

using (17) and equality $\hat{\chi}_e = \Delta\chi_e + \chi_e$, yields to

$$\dot{V} = -V_0 - \lambda_e \int_0^1 (\Delta\chi_e)^2 dx - \lambda_s \int_0^1 (\Delta S)^2 dx$$

$$+ \int_0^1 \Delta\chi_e f_e dx + \int_0^1 \Delta\chi_e g_e dx + \int_0^1 \Delta S f_s dx \qquad (18)$$

$$- \int_0^1 \Delta S g_s dx.$$

Let us investigate (18) term by term, first the terms involving χ_e:

$$\int_0^1 \Delta\chi_e f_e dx + \int_0^1 \Delta\chi_e g_e dx =$$

$$\lambda_e \int_0^1 \chi_e \left(\chi_e + \frac{\dot{\chi}_e}{\sigma_e}\right) dx$$

$$+ \int_0^1 \hat{\chi}_e f_e dx - \frac{\mu_e^2}{\mu_e \|\hat{\chi}_e\|_2 + \epsilon_e} \|\hat{\chi}_e\|_2^2 \qquad (19)$$

$$+ \int_0^1 \frac{\mu_e^2}{\mu_e \|\hat{\chi}_e\|_2 + \epsilon_e} \|\hat{\chi}_e\|_2 \|\chi_e\|_2 dx$$

$$\leq \lambda_e \|\chi_e\|_2^2 + \frac{1}{\vartheta_1} \|\chi_e\|_2 \|\dot{\chi}_e\|_2 + \epsilon_e + \mu_e \|\chi_e\|_2$$

and now, the terms in S:

$$\int_0^1 \Delta S f_s dx - \int_0^1 \Delta S g_s dx = \lambda_s \int_0^1 S \left(S + \frac{\dot{S}}{\sigma_s}\right) dx$$

$$+ \int_0^1 \hat{S} f_s dx - \frac{\mu_s^2}{\mu_s \|\hat{S}\|_2 + \epsilon_s} \|\hat{S}\|_2^2 \qquad (20)$$

$$+ \int_0^1 \frac{\mu_s^2}{\mu_s \|\hat{S}\|_2 + \epsilon_s} \|\hat{S}\|_2 \|S\|_2 dx$$

$$\leq \lambda_s \|S\|_2^2 + \frac{1}{\vartheta_2} \|S\|_2 \|\dot{S}\|_2 + \epsilon_s + \mu_s \|S\|_2.$$

Replacing (19) and (20) in (18), we get the following inequality for the Lyapunov-like function (10):

$$\dot{V} \leq -V_0 - \lambda_e \int_0^1 (\Delta\chi_e)^2 dx - \lambda_s \int_0^1 (\Delta S)^2 dx + v(t), \quad (21)$$

where $v(t) = \lambda_e \|\chi_e\|_2^2 + \frac{1}{\vartheta_1} \|\chi_e\|_2 \|\dot{\chi}_e\|_2 + \epsilon_e + \mu_e \|\chi_e\|_2 + \lambda_s \|S\|_2^2 + \frac{1}{\vartheta_2} \|S\|_2 \|\dot{S}\|_2 + \epsilon_s + \mu_s \|S\|_2$.

Since $\chi_e(x,.)$, $\dot{\chi}_e(x,.)$, $S(x,.)$ and $\dot{S}(x,.)$ are L^2-bounded, $v(t)$ is bounded and thus ΔT, $\Delta\chi_e$ and ΔS are ultimately uniformly L^2-bounded, the L^2-norm of these variables converge to the region defined by $v(t)$. By making ϵ_e, ϵ_s, σ_e and σ_s sufficiently small and ϑ_1 and ϑ_2 sufficiently large, $v(t)$ can be arbitrary very small.

In addition, if the plant is persistently exciting, a uniform boundedness region for the parameters estimation error can be made arbitrary small by a convenient choice of the estimators parameter gains [7].

V. SIMULATION RESULTS

Simulation with computed data is carried out to evaluate the reconstruction performance of the spatially varying input/diffusion adaptive identifier (6),(7). The dissipation parameter τ is assumed to be known and constant. Simulations are performed using MATLAB/Simulink.

Since the identifier is infinite-dimensional, the bspline-cubic

Galerkin method is used in order to implement it ([3], [1]). The simulated data is generated by using:

$$\begin{cases} \chi_e(x,t) = (0.1 + 5\,x + 2\,x^2 + 4\,x^3)\,\mathbf{1}(t); \ \tau = 0.05 \\ S(x,t) = \dfrac{10^5}{\sqrt{2\,\pi\,\sigma}}\exp\left(\dfrac{-(x-\mu)^2}{2\,\sigma^2}\right)\mathbf{1}(t) \\ x \in [0,1], \quad t \in [0,1], \quad dx = 0.05, \quad dt = 0.01. \end{cases} \quad (22)$$

The choice of χ_e, τ and S is motivated by the example proposed by [13], where it was assumed that the diffusion coefficient has a monomial monotonically increasing function and the heating source undergoes a spatial Gaussian form. These parameters were considered constant in time. Fig. 1 presents the space variations of χ_e and S used in the mock-up data to generate T.
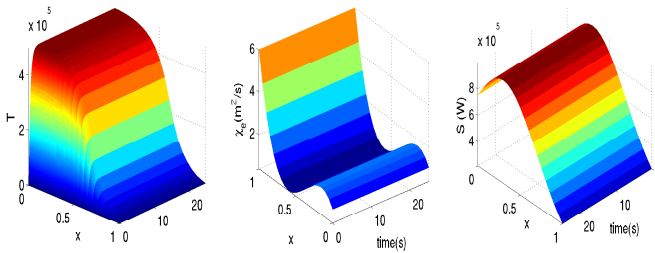

Fig. 1: T, χ_e and S for the mock-up example

To evaluate the reconstruction performance using the adaptive identifier (6),(7), the initialization of the filter was arbitrary. The choice of the bases dimensions in the Galerkin formulation is as follows. For χ_e and S, we have chosen a dimension of $n = 9$, whereas for T, we have chosen $n = 20$ for its space basis dimension. It is a good trade-off between precision and convergence rate as shown in Fig. 2. Note that the space basis dimension of T is related to the number of required sensors as we are using the bsplines-Galerkin method. In practice (for the tokamak facility) more sensors are available.
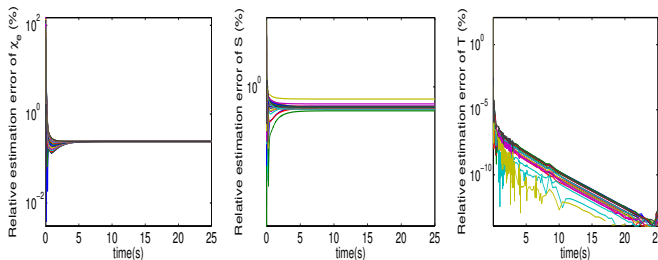

Fig. 2: Estimation errors of χ_e, S, and T for the mock-up example

From Fig. 2, the estimation of χ_e, S and T using the adaptive identifier (6),(7) is satisfactory. This figure shows the time evolution of each node relative estimation error. The filter needs few iterations to converge to the original

variables. The choice of ϑ_i, $i = 0, 1, 2$ is crucial. From the simulation, we observe that for each ϑ_0 corresponds a couple of $(\vartheta_1, \vartheta_2)$ and increasing these estimation/adaptation gains leads to faster state and parameters (χ_e and S) convergence. However, similarly to the gradient search method, beyond some points, larger sizes lead to oscillations and even slow convergence.

VI. Conclusion

In this paper we have studied and tested state input and parameter adaptive estimation for a linear parabolic PDE representing heat transport. Two related problems were considered. First, only space-varying parameter/input were considered. The proposed identifier tested in simulation gives good results. For distributed space-time varying input/parameter with unknown bounds, only theoretical results were given. More simulations on computed and real data are needed to establish the performance of the proposed technique.

In comparison with our previous results using a Kalman filter in the finite dimensional framework, the present approach provided a clearer analytical framework but is more sensitive to tuning parameters. In this paper, noise measurement implication is not investigated. In our future works, this question will be addressed.

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