Remote Output Stabilization Under Two Channels Time-Varying Delays

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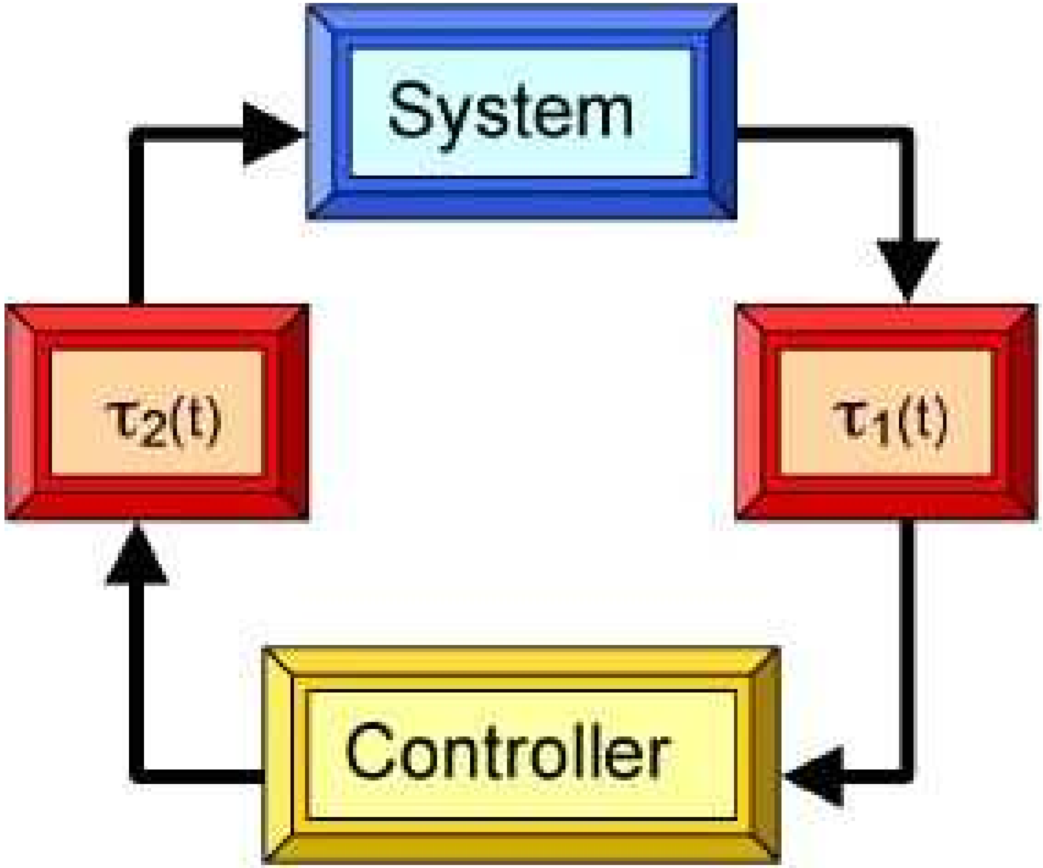


System
τ2(t)
τ1(t)
Controller

Contents

I. Background on Time-Delay

- Lyapunov-Krasovskii approaches (Niculescu, Kharitonov, Verriest, Yu...): constant time-delays/upper bound.
- Passivity (Anderson/Spong, Niemeyer/Slotine...): teleoperation.
- Stability (Bo Lincoln 03, Meinsma/Zwart, Sename...): robustness.
- Pole-placement (Kwon/Pearson, Manitius/Olbrot...)
- Stochastic approach (Nilsson 98...): LQG control.

II. Problem Formulation

Consider the system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t - \tau_2(t)) && (1) \\ y(t) &= Cx(t) && (2) \\ \dot{z}_i(t) &= W_i(z_i, t) + F_i(z_i, t)\tau_{diz}(t), && (3) \\ \tau_i(t) &= E_i(z_i, t) + R_i(z_i, t)\tau_{di\tau}(t), \ \ i = 1, 2 && (4) \end{aligned}$$

(3)-(4) satisfy

$$\tau_{max} \geq \tau(t) \geq 0 \quad \forall t \geq 0$$

$$\exists t_0 \geq 0 \quad \text{s.t.} \quad |\tau(t)| < 1 \quad \forall t \geq t_0$$

III. Control design

- Assume that the delayed state or output is measurable, (i.e. $x(t-\tau_1(t))$ or $y(t-\tau_1(t))$),
- Define $v(t) = u(t-\tau_2(t))$ and $\infty > \delta(t) \geq 0$,
- (1) $\rightarrow \dot{x}(t+\delta(t)) = Ax(t+\delta(t)) + Bv(t+\delta(t))$,
- Assume that $v(t+\delta(t)) = -Kx(t+\delta(t))$.

Then the resulting closed-loop equation is

$$\dot{x}(t+\delta(t)) = (A-BK)x(t+\delta(t)) = A_{cl}x(t+\delta(t))$$

- Two conditions have to be satisfied:

  1. The possibility to predict

$$x(t+\delta) = e^{A(\delta+\tau_1)}x(t-\tau_1) + e^{A(t+\delta)} \int_{t-\tau_1}^{t+\delta} e^{-A\theta} Bu(\theta - \tau_2(\theta))d\theta$$

  2. The possibility to assign $v(t+\delta(t))$.

- From causality:

$$\delta(t) \doteq max \{\delta \geq 0 \ \mid \ \forall\theta \in [t-\tau_1, t+\delta], \quad \theta - \tau_2(\theta) \leq t\} \qquad (5)$$

- Both $1.$ and $2.$ are satisfied if $t+\delta(t) - \tau_2(t+\delta(t)) = t$, and

$$u(t) = v(t+\delta(t)) = -Kx(t+\delta(t))$$

- The resulting control is

$$\begin{aligned} u(t) &= -Ke^{A(\delta+\tau_1)}x(t-\tau_1) - Ke^{A(t+\delta)}\int_{t-\tau_1}^{t+\delta} e^{-A\theta}Bu(\theta-\tau_2(\theta))d\theta \\ \delta(t) &= \tau_2(t+\delta(t)) \end{aligned}$$

i.e. if

$$\dot{z}(t) = A_t z(t) + B_t v(t), \quad with \;\; z(0) = z_0 \tag{6}$$

$$\tau_2(t) = C_t z(t) \tag{7}$$

then

$$\delta(t) = C_t\left[e^{A_t\delta}z(t) + e^{A_t(t+\delta)}\int_t^{t+\delta} e^{-A_t\theta}B_t v(\theta)d\theta\right]$$

Stability analysis

- **Lemma:** Consider the system

$$\frac{dx}{dt}(t+\delta(t)) = A_{cl}x(t+\delta(t))$$

for $t \geq 0$ and $\delta(0) = \delta_0$. If the following conditions hold:

$i)$ the eigenvalues of A_{cl} are in the open LHP,

$ii)$ $\infty > \delta_M \geq \delta(t) \geq 0$,

$iii)$ $||\dot{\delta}(t)|| < 1, \quad \forall t \geq \delta_0$.

then $\lim_{t\to\infty} ||x(t+\delta(t))|| = 0, \ \forall t \geq \delta_0$ and for all bounded values of $x(\delta_0)$.

Theorem 1. *Assume that the delay dynamics(3)-(4) is such that the following holds for* $\tau_i(t)$, $i = 1, 2$, $t \geq t_0$:

$A1)$ $\infty > \tau_i^{max} \geq \tau_i(t) \geq 0,$
$A2)$ $|\dot{\tau}_2| < 1 \quad \forall t \geq t_0,$

Then, the feedback control law

$$\begin{aligned} u(t) &= -Ke^{A(\delta+\tau_1)}x(t-\tau_1) - Ke^{A(t+\delta)} \int_{t-\tau_1}^{t+\delta} e^{-A\theta} Bu(\theta - \tau_2(\theta)) d\theta \\ \delta(t) &= \tau_2(t + \delta(t)) \end{aligned}$$

ensures that the closed-loop system is bounded, and that the state $x(t)$ *converges exponentially to zero.*

IV. Observer-Based Control

– Luenberger state-observer:

$$\begin{aligned}\dot{\hat{x}}(t-\tau_1) &= A\hat{x}(t-\tau_1) + Bu(t-\tau_1-\tau_2(t-\tau_1)) \\ &\quad + H\{y(t-\tau_1) - C\hat{x}(t-\tau_1)\}\end{aligned}$$

– The resulting observation error

$$\epsilon(t-\tau_1) \doteq x(t-\tau_1) - \hat{x}(t-\tau_1)$$

– The complete closed-loop dynamics is:

$$\begin{aligned}\dot{x}(t+\delta(t)) &= A_{cl}x(t+\delta(t)) + BKe^{A(\delta+\tau_1)}\epsilon(t-\tau_1) \\ \dot{\epsilon}(t-\tau_1) &= \hat{A}_{cl}\epsilon(t-\tau_1)\end{aligned}$$

$\Rightarrow$ Exponentially stable if τ_i bounded and $|\dot{\tau}_i| < 1$ for $t > t_0$.

Example: remote output stabilisation

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t - \tau_2)$$

$$y = [1 \;\; 0]x$$

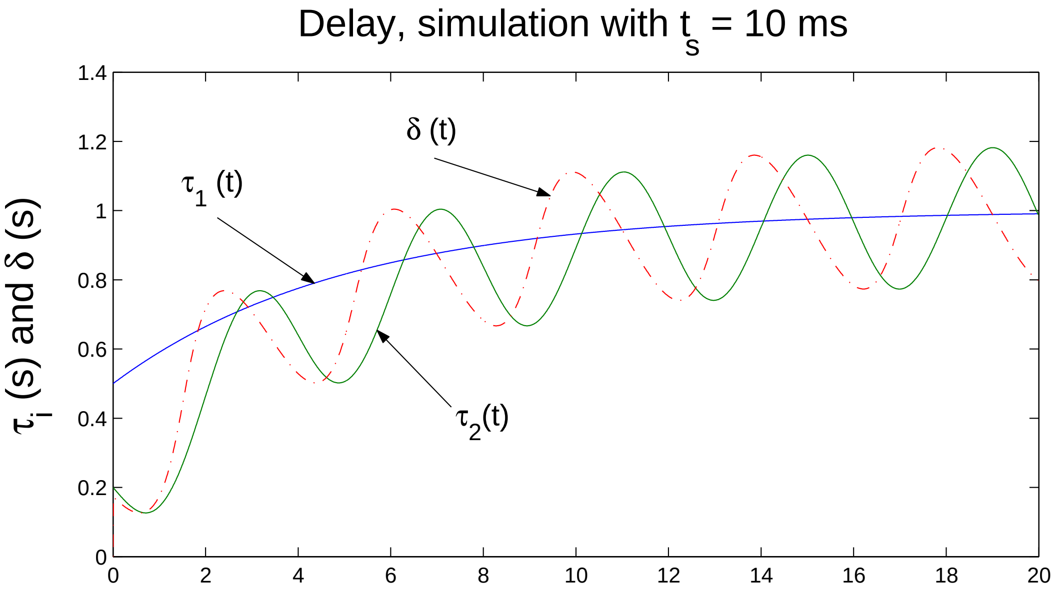
with $K = [-2 \quad -1]$ and $x_0 = [1 \;\; 0.5]^T$.

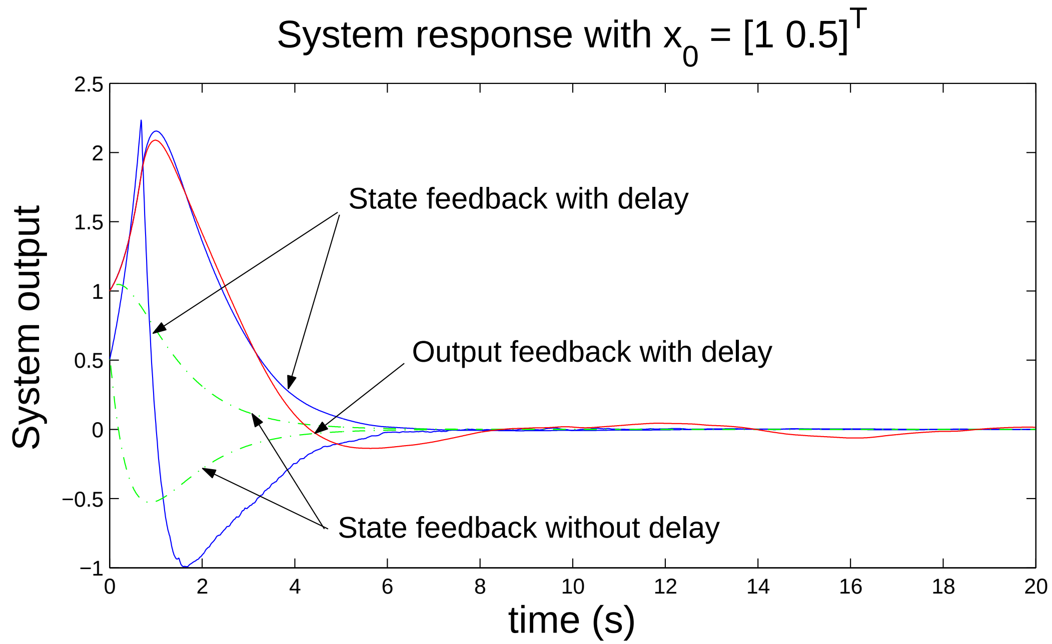
$$\dot{z}_1 = -\frac{1}{5} z_1 + \frac{2}{5}, \quad z_1(0) = 1$$

$$\tau_1 = z_1$$

$$\dot{z}_2 = -\frac{1}{2} z_2 + 1, \quad z_2(0) = 4$$

$$\tau_2 = z_2 - \frac{1}{2} sin(\frac{1}{2}\pi t)$$





Conclusions

– Remote stabilization via communication networks
  $\Rightarrow$ stabilizing an open-loop unstable system with $\tau(t)$.
– The proposed controller:
  * based on a $\delta(t)$-step ahead predictor,
  * results in an exponentially converging closed-loop system and pole placement on the time-shifted system,
  * applied to remote output stabilization.
– Open problems:
  * use of state-dependant models,
  * computation of δ.

Proof of Stability Analysis

– Introduce $\zeta(t) = t + \delta(t)$, then

$$\frac{dx(\zeta)}{d\zeta} = \gamma(t) A_{cl} x(\zeta), \quad \gamma(t) = \frac{1}{1 + \frac{d\delta(t)}{dt}}$$

– Consider $\qquad V(\zeta) = x(\zeta)^T P x(\zeta)$

– with (i) and (iii), $\exists \quad Q > 0$ such that

$$\frac{dV(\zeta)}{d\zeta} \leq -\lambda_m(Q) ||x(\zeta)||^2 \gamma(t) < 0$$

– Using the bounds of $V(\zeta)$, and integrating from $\zeta(0) = \delta_0$ to $\zeta(t)$:

$$V(\zeta) \leq V(\delta_0)e^{-\Psi(\delta_0,\zeta)} \Rightarrow ||x(\zeta)||^2 \leq \frac{\lambda_m(P)}{\lambda_M(P)}||x(\delta_0)||^2 e^{-\Psi(\delta_0\zeta)}$$

where (iii)

$$\Psi(\delta_0,\zeta) = \frac{\lambda_m(Q)}{\lambda_M(P)}\int_{\delta_0}^{\zeta}\gamma(\theta)d\theta \geq \frac{\lambda_m(Q)}{\lambda_M(P)}\frac{1}{2}(\zeta - \delta_0),$$

– from (ii), $t \to \infty$ implies $\zeta \to \infty$ as well, and

$$\lim_{t\to\infty}\Psi(\delta_0,\zeta) = +\infty$$

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Existence of

$$\delta(t) \doteq max \{\delta \geq 0 \quad | \quad \forall \theta \in [t, t+\delta], \quad \theta - \tau(\theta) \leq t\}$$

- **Proposition:** The optimization problem (5) admits a solution $\delta(t)$ for which

$$\delta(t) - \tau(t + \delta(t)) = 0$$

- **Proof:** Let t be some given instant, the admissible set for (5) is:
  * non empty since $\delta = 0$: $t - \tau(t) \leq t$,
  * bound by $\delta_{sup} = \sup_{\sigma \geq 0} \tau(\sigma)$,
  * closed by classical continuity arguments.

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- **Corollary:** The control law applied to the system (1)-(2), has a bounded solution and exponentially converges to zero, for all $t \geq \delta_0$.

- **Proof:**
  - (1)-(2) linear: its states cannot diverge in finite time,
  - $\forall \delta_0$ bounded, $x(\delta_0)$ bounded,
  - from the previous lemma, the state then exponentially converge to zero.

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Theorem 2. *Assume that the delay dynamics(3)-(4) is such that the following holds for* $\tau_i(t), i = 1, 2.$,

A1) $\infty > \tau_i(t) \geq 0 \quad \forall t,$

A2) $|\dot{\tau}_i| < 1 \quad \forall t > t_0$

Then, the observer-based feedback control law

$$
\begin{aligned}
u(t) &= -Ke^{A(\delta(t)+\tau_1(t))}\hat{\hat{x}}(t) - Ke^{A(t+\delta(t))}\int_{t-\tau_1(t)}^{t+\delta(t)} e^{-A\theta}Bu(\theta - \tau_2(\theta))d\theta \\
\dot{\hat{\hat{x}}}(t) &= A\hat{\hat{x}}(t) + Bu(t - \tau_1 - \tau_2(t - \tau_1)) + H\{y(t - \tau_1) - C\hat{\hat{x}}(t)\}
\end{aligned}
$$

with $\hat{\hat{x}}(t) \doteq \hat{x}(t - \tau_1(t))$ *ensures that the closed-loop system is bounded, and that the state* $x(t)$ *converges exponentially to zero.*