Methods for analysis and control of dynamical systems
Lecture 2: Modelling of dynamical systems

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Introduction

Methods for system modelling

Physical examples
  Hydraulic tanks
  Satellite attitude control model
  The DVD player
  The suspension system
  The wind tunnel
  Energy and comfort management in intelligent building

State space representation
  Physical examples
  Linearisation
  Conversion to transfer function
References

Some interesting books:

Types of models

- **Finite state models** (Petri nets, Grafcet) of logical systems: discrete-event systems
- **Graph models**: Bond graph. Allow a physical description in a unique way whatever the physical domain is.
- **Experimental models**: allow to reproduce an input-output behavior.
- **State models**: A mathematical description of the system in terms of a minimum set of variables \( x_i(t), i = 1, \ldots, n \), together with knowledge of those variables at an initial time \( t_0 \) and the system inputs for time \( t \geq t_0 \), are sufficient to predict the future system state and outputs for all time \( t \geq t_0 \).
Identification based method- Black box models

- System excitations using step inputs, sinusoidal signals, or PRBS (Pseudo Random Binary Signal)
- Determination of a transfer function reproducing the input/output system behavior
- Method: direct identification (Strejc) or by optimization.
- Objective: determination of the set of model parameters.
Different issues for modelling (2)

Knowledge-based method - White box models

- Represent the system behavior using differential and/or algebraic equations, based on physical knowledge.
- Formulate a nonlinear state-space model, i.e. a matrix differential equation of order 1.
- Determine the steady-state operating point about which to linearize.
- Introduce deviation variables and linearize the model.
Why knowledge-based method are of interest?

- Dynamical systems where physical equations can be derived: electrical engineering, mechanical engineering, aerospace engineering, Microsystems, process plants ....
- Include physical parameters: easy to use when parameters are changed for design
- State variables have physical meaning.
- Allow for including non-linearities (state constraints)
- Easy to extend to Multi-Input Multi-Output (MIMO) systems
- Advanced control design methods are based on state space equations (reliable numerical optimisation tools)
Height control of a single Tank

Consider a water tank of area $S$, height $H$, fed by an input flow $Q_e$, with an output flow $Q_s$

![Diagram of water tank with input and output flows]

**Figure:** Bac.

Usually the flow is considered to be proportional to the square root of the pressure difference, then

$$Q_s = k_t \sqrt{H}$$

and

$$S \frac{dH}{dt} = Q_e - Q_s = Q_e - k_t \sqrt{H}$$
A satellite attitude control model

A simple model for a one-axis system is:

\[ I \ddot{\theta} = M_D + F_c d \]

where \( I \) is the inertia, \( \theta \) the angular position, \( M_D \) a small disturbance moment on the satellite, \( F_c \) the control force that comes from the reaction jets, \( d \) the distance from the jet to the center of gravity.
the DVD player

Spindle Motor

Objective moving lens

Optical Pick-up Unit (OPU)

Sledge rails

Sledge

Transmission

Sledge Motor
Use of physical principles. Focus and radial actuators: are constituted by a lens attached to the pick-up body by two parallel leaf spring, and moved in vertical and radial direction by a voice coil and a magnet.
Modelling the DVD player (2)

- Actuator: voltage \( v(t) \). Controls the pick-up voice coil
- Output signal: laser spot position \( x(t) \)

**Electrical part:** the voltage \( v(t) \) applied to the R-L circuit makes flow in it a current \( i(t) \):

\[
L \frac{\partial i(t)}{\partial t} + Ri(t) = v(t) - K_e \frac{\partial x(t)}{\partial t}
\]  
(1)

**Magnetic part:**

\[
f(t) = K_e i(t)
\]  
(2)

where \( K_e \) is the back-emf constant,

**Mechanical part:** The force \( f(t) \) [N] acts on the objective lens mass \( M \) [Kg], making the actuator moves:

\[
M \frac{\partial^2 x(t)}{\partial t^2} + D \frac{\partial x(t)}{\partial t} + kx(t) = f(t)
\]  
(3)
Data of the DVD player 1

Table: Values of the physical parameters of Pick-up 1 (for focus and tracking actuators), from Pioneer.

<table>
<thead>
<tr>
<th>Name</th>
<th>Description</th>
<th>Value Focus</th>
<th>Value Tracking</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>DC resistance of coil</td>
<td>$5.4 \pm 1.1 \ \Omega$</td>
<td>$5.9 \pm 1.2 \ \Omega$</td>
</tr>
<tr>
<td>L</td>
<td>Inductance of coil</td>
<td>$15 \pm 6 \ \mu H$</td>
<td>$9 \pm 6 \ \mu H$</td>
</tr>
<tr>
<td>M</td>
<td>Moving mass</td>
<td>$0.7 \ \text{g}$</td>
<td>$0.7 \ \text{g}$</td>
</tr>
<tr>
<td>$S_{DC}$</td>
<td>DC Sensitivity</td>
<td>$2.69 \ \text{mm/V}$</td>
<td>$0.63 \ \text{mm/V}$</td>
</tr>
<tr>
<td>$f_0$</td>
<td>Resonance frequency</td>
<td>$30 \pm 7 \ \text{Hz}$</td>
<td>$47 \pm 7 \ \text{Hz}$</td>
</tr>
<tr>
<td>$Q_{dB}$</td>
<td>Resonance peak</td>
<td>$\leq 15 \ \text{dB}$</td>
<td>$\leq 15 \ \text{dB}$</td>
</tr>
</tbody>
</table>
Data of the DVD player 2

Table: Values of the physical parameters of Pick-up 2 (for focus and tracking actuators), from Sanyo

<table>
<thead>
<tr>
<th>Name</th>
<th>Description</th>
<th>Value Focus</th>
<th>Value Tracking</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>DC resistance of coil</td>
<td>6.5 ± 1 Ω</td>
<td>6.5 ± 1 Ω</td>
</tr>
<tr>
<td>L</td>
<td>Inductance of coil</td>
<td>25 ± 6 μH</td>
<td>18 ± 6 μH</td>
</tr>
<tr>
<td>M</td>
<td>Moving mass</td>
<td>0.33 g</td>
<td>0.33 g</td>
</tr>
<tr>
<td>$S_{DC}$</td>
<td>DC Sensitivity</td>
<td>0.94 mm/V</td>
<td>0.27 mm/V</td>
</tr>
<tr>
<td>$f_0$</td>
<td>Resonance frequency</td>
<td>52 ± 7 Hz</td>
<td>52 ± 7 Hz</td>
</tr>
<tr>
<td>$Q_{dB}$</td>
<td>Resonance peak</td>
<td>$\leq 20$ dB</td>
<td>$\leq 20$ dB</td>
</tr>
</tbody>
</table>
Control of dynamical systems

O.Sename

Introduction

Methods for system modelling

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Energy and comfort management in intelligent building

State space representation

Physical examples
Linearisation
Conversion to transfer function

Calculation of the parameters

The elastic constant \( k \) \([N/m]\), the dumping factor \( D \) \([Ns/m]\), and the electro-magnetic constant \( K_e \) \([Wb/m]\) are:

\[
\begin{align*}
   \omega_n &= 2\pi f_0 \\
   k &= M \times \omega_n^2 \\
   D &= \omega_n \times M \times \sqrt{2 \left( 1 - \sqrt{1 - \frac{1}{Q^2}} \right)} \\
   K_e &= kRS_{DC}
\end{align*}
\]

where \( Q \) denotes the absolute value of the actuator amplitude peak, at the resonance frequency \( f_0 \), \( S_{DC} \) \([mm/V]\) is the value of the actuator DC sensitivity, \( M \) \([kg]\) is the objective lens mass \( R \) \([\Omega]\) and \( L \) \([H]\) the resistance and inductance of the coil.
Calculation of the transfer function

**Electrical part:**

\[ I(s) = \frac{1}{Ls + R} [V(s) - K_e s X(s)] \]  \hspace{1cm} (4)

**Magnetic part:**

\[ F(s) = K_e I(s) \]  \hspace{1cm} (5)

**Mechanical part:**

\[ \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Ds + k} \]  \hspace{1cm} (6)

Combining these equations, it leads

\[ H(s) = \frac{X(s)}{V(s)} = \frac{K_e}{s^3 + \left(\frac{R}{L} + \frac{D}{M}\right) s^2 + \left(\frac{DR}{ML} + \frac{k}{M} + \frac{K_e^2}{ML}\right) s + \frac{kR}{ML}} \]  \hspace{1cm} (7)
A simplified quarter vehicle model with semi-active suspension.

\[ \begin{align*}
    z_s & \quad \text{(relative position of the chassis and wheel)} \\
    m_s & \quad \text{(mass of the chassis)} \\
    m_{us} & \quad \text{(mass of the wheel)} \\
    k_s & \quad \text{(spring coefficient of the suspension)} \\
    k_t & \quad \text{(spring coefficient of the tire)} \\
    u & \quad \text{(active damper force)} \\
    z_r & \quad \text{(road profile)}
\end{align*} \]
Suspension system (2)

The mechanical equations are:

\[
\begin{align*}
    m_s \ddot{z}_s &= -F_k(z_{\text{def}}) - F_c(\dot{z}_{\text{def}}) \\
    m_{us} \ddot{z}_{us} &= F_k(z_{\text{def}}) + F_c(\dot{z}_{\text{def}}) - k_t(z_{us} - z_r)
\end{align*}
\]

(8)

where \( F_k(z_{\text{def}}) \) and \( F_c(\dot{z}_{\text{def}}) \) (with \( z_{\text{def}} = z_s - z_{us} \) and \( \dot{z}_{\text{def}} = \dot{z}_s - \dot{z}_{us} \)) are the nonlinear forces provided by the spring and damper respectively.

Figure: Nonlinear forces provided by the Spring (left) and the Damper (right).
The involved model parameters have been identified on a "Renault Mégane Coupé" car and are given below.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_s$</td>
<td>315 kg</td>
<td>sprung mass</td>
</tr>
<tr>
<td>$m_{us}$</td>
<td>37.5 kg</td>
<td>unsprung mass</td>
</tr>
<tr>
<td>$k$</td>
<td>29500 N/m</td>
<td>suspension linearized stiffness</td>
</tr>
<tr>
<td>$c$</td>
<td>1500 N/m/s</td>
<td>suspension linearized damping</td>
</tr>
<tr>
<td>$k_p$</td>
<td>210000 N/m</td>
<td>tire stiffness</td>
</tr>
<tr>
<td>$[Z_{def}, \bar{Z}_{def}]$</td>
<td>$[-8, 6]$ cm</td>
<td>suspension’s deflection limits</td>
</tr>
</tbody>
</table>

**Table:** Parameters model of a "Renault Mégane Coupé".
Objective: feedback control of the Mach number in a wind tunnel (NASA)

In steady-state operating conditions (some constant fan speed, liquid nitrogen injection rate, and gaseous-nitrogen vent rate), the dynamic response of the Mach number perturbations $\delta M$ to small perturbations in the guide vane angle actuator $\delta \theta_A$

$$\tau \delta \dot{M}(t) + \delta M(t) = k \delta \theta(t - h)$$
$$\delta \dot{\theta}(t) + 2\xi \omega \delta \dot{\theta}(t) + \omega^2 \delta \theta(t) = \omega^2 \delta \theta_A(t)$$

$\delta \theta(t)$ is the guide vane angle.

Time-delay $h$: transportation time between the guide vanes of the fan and the test section of the tunnel. $h$ varies as a function of the temperature and is such that $0.288 \leq h \geq 0.455$ s.
Some issues

Why intelligent control systems (Energy Management System)?

- Use several actuators: lights, window opening, shading, heating/cooling (air conditioning)...
- Control objectives:
  - Air quality: CO$_2$, particule matter, Volatile Organic Compounds
  - Comfort: humidity, temperature, luminance
  - Energy savings: consumption

Heating Ventilating and Air Conditioning (HVAC)

A system complex to be modelled:

- A Multi-Zone system
- Wireless Sensor Network
- Air flow (thermodynamics: fans, ducts, doors,.. ) and Thermal models (temperature, humidity)
L’équation fondamentale représentant la variation de température de part et d’autres d’un mur est la suivante:

\[ \rho . c_v . V \frac{dT}{dt} = \frac{\lambda_{wall}}{\Delta x} . A . (T_{out} - T) + \dot{Q}_{sources} \] (9)

où

<table>
<thead>
<tr>
<th>(T) (K)</th>
<th>température de la pièce</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T_{out}) (K)</td>
<td>température extérieure</td>
</tr>
<tr>
<td>(c_v) (J/kgK)</td>
<td>capacité thermique de l’air à volume constant = 719</td>
</tr>
<tr>
<td>(c_p) (J/kgK)</td>
<td>capacité thermique de l’air à pression constante = 1010</td>
</tr>
<tr>
<td>(V) (m³)</td>
<td>Volume de la pièce</td>
</tr>
<tr>
<td>(\rho) (kg/m³)</td>
<td>densité de l’air = 1.169</td>
</tr>
<tr>
<td>(\Delta x) (m)</td>
<td>épaisseur</td>
</tr>
<tr>
<td>(A) (m²)</td>
<td>Surface du mur</td>
</tr>
<tr>
<td>(\dot{Q}_{sources})</td>
<td>Sources (extérieures + contrôle)</td>
</tr>
</tbody>
</table>

Les coefficients de conductivité \(\lambda_{wall}\) dépendent des matériaux composant les murs, et sont donnés :
Many dynamical systems can be represented by Ordinary Differential Equations (ODE) as

\[
\begin{align*}
\dot{x}(t) &= f((x(t), u(t), t), \quad x(0) = x_0 \\
y(t) &= g((x(t), u(t), t))
\end{align*}
\] (10)

where \( f \) and \( g \) are non linear functions.
Definition of state space representations

A **continuous-time** LINEAR state space system is given as:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0 \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

- \(x(t) \in \mathbb{R}^n\) is the system state (vector of state variables),
- \(u(t) \in \mathbb{R}^m\) the control input
- \(y(t) \in \mathbb{R}^p\) the measured output
- \(A, B, C\) and \(D\) are real matrices of appropriate dimensions
- \(x_0\) is the initial condition.

\(n\) is the order of the state space representation.

**Matlab**: \(ss(A,B,C,D)\) creates a SS object \(SYS\) representing a continuous-time state-space model.
Height control of a single Tank

In steady state: \( Q_e = Q_0, \ H = H_0 \)

Consider the variations \( q_e, q_s, h \) around the steady state as:

\( Q_e = Q_0 + q_e; \quad Q_s = Q_0 + q_s; \quad H = H_0 + h. \)

This leads to the equation:

\[
S \frac{dh}{dt} = q_e - k_t (\sqrt{H_0 + h} - \sqrt{H_0})
\]

Using the first order approximation \((1 + x)^\alpha = 1 + \alpha x\), it leads

\[
S \frac{dh}{dt} = q_e h
\]

Denoting the state variable \( x = h \), the control input \( u = q_e \), the output \( y = h \), we get

\[
\dot{x} = Ax + Bu \quad (12)
\]

\[
y = Cx \quad (13)
\]

with \( A = -\frac{k_t}{2\sqrt{H_0}}, \quad B = \frac{1}{S} \) and \( C = 1 \).
Some examples

**Suspension system**
Choose the state variables and give the state space representation of the system, with input $z_r$ (not controlled) and output $z_s - z_{us}$ or $\ddot{z}_s$.

**Satellite**
Choose the state variables and give the state space representation of the system, with controlled input $F_c$, disturbance input $M_D$ and output $\theta$. 
A wind tunnel

In steady-state operating conditions (fan speed, liquid nitrogen injection rate and gaseous-nitrogen vent rate) the dynamic response of the Mach number is given by the following system:

\[
\dot{x}(t) = \begin{bmatrix}
-0.5091 & 0 & 0 \\
0 & 0 & 1 \\
0 & -36 & -9.6
\end{bmatrix} x(t) + \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} x(t-h) + \begin{bmatrix}
0.005956 \ 0 \\
0 \ 0 \\
36
\end{bmatrix} u(t) + \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} w(t)
\]

\[
y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t) + w(t)
\]

\[
z(t) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x(t)
\]

\[x(t) = \phi(t); \quad t \in [-h, 0]\]

where \( h = 0.33 \text{sec.} \), \( x_1 \) is the Mach number, \( x_2 \) is the guide vane angle and \( x_3 = \dot{x}_2 \).
Example : Wind turbine

![Wind turbine image]

An complete model (ADAMS) includes 193 DOFs.
Some important issues

- A complete ADAMS model includes 193 DOFs to represent fully flexible tower, drive-train, and blade components ⇒ simulation model
- Different operating conditions according to the wind speed
- Control objectives: maximize power, enhance damping in the first drive train torsion mode, design a smooth transition different modes
- A Generator torque controller to enhance drive train torsion damping in Regions 2 and 3
- The control model is obtained by linearisation of a non linear electro-mechanical model:

\[
\begin{cases}
\dot{x}(t) = Ax(t) + Bu(t) + Ed(t) \\
y(t) = Cx(t)
\end{cases}
\]

where \(x_1 = \) rotor-speed \(x_2 = \) drive-train torsion spring force, \(x_3 = \) rotational generator speed
\(u = \) generator torque, \(d \) : wind speed
More generally

Reformulate Nth-order differential equation into N simultaneous first-order differential equations

\[
\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \dot{y} + a_0 y = f
\]

Define the state variables:

\[
\begin{align*}
    x_1 &= \ldots, \\
    x_2 &= \ldots, \\
    \ldots x_n &= \ldots,
\end{align*}
\]

and give the according state space representation. 

**Remark:** Knowledge of state variables allows one to determine every possible output of the system.
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State space representation

Physical examples

Linearisation

Conversion to transfer function

Linearisation
Linearisation

The linearisation can be done around an equilibrium point or around a particular point defined by:

\[
\begin{align*}
\dot{x}_{eq}(t) &= f((x_{eq}(t), u_{eq}(t), t), \text{ given } x_{eq}(0) \\
y_{eq}(t) &= g((x_{eq}(t), u_{eq}(t), t)
\end{align*}
\]  

(14)

Defining

\[
\tilde{x} = x - x_{eq}, \quad \tilde{u} = u - u_{eq}, \quad \tilde{y} = y - y_{eq}
\]

this leads to a linear state space representation of the system, around the equilibrium point:

\[
\begin{align*}
\dot{\tilde{x}}(t) &= A\tilde{x}(t) + B\tilde{u}(t), \\
\tilde{y}(t) &= C\tilde{x}(t) + D\tilde{u}(t)
\end{align*}
\]  

(15)

with \( A = \frac{\partial f}{\partial x} \mid_{x=x_{eq}, u=u_{eq}}, B = \frac{\partial f}{\partial u} \mid_{x=x_{eq}, u=u_{eq}}, C = \frac{\partial g}{\partial x} \mid_{x=x_{eq}, u=u_{eq}} \) and \( D = \frac{\partial g}{\partial u} \mid_{x=x_{eq}, u=u_{eq}} \)

Usual case

Usually an equilibrium point satisfies:

\[
0 = f((x_{eq}(t), u_{eq}(t), t)
\]  

(16)

For the pendulum, we can choose \( y = \theta = f = 0 \).
Linear systems: transfer function
Consider a linear system given by:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0 \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]  

(17)

Using the Laplace transform (and assuming zero initial condition \(x_0 = 0\)), (17) becomes:

\[
sx(s) = Ax(s) + Bu(s) \quad \Rightarrow \quad (sI_n - A)x(s) = Bu(s)
\]

Then the transfer function matrix of system (17) is given by

\[
G(s) = C(sI_n - A)^{-1}B + D = \frac{N(s)}{D(s)}
\]

(18)

**Matlab:** if SYS is an SS object, then `tf(SYS)` gives the associated transfer matrix. Equivalent to `tf(N, D)`