Methods for analysis and control of dynamical systems
Lecture 3: tools for performance analysis

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Outline

- Introduction
- Time-domain performance criteria
- Frequency-domain performance criteria
- Sensitivity function analysis
- Towards robustness analysis
Goodwin et al, 01: “Success in control engineering depends on taking a holistic viewpoint. Some of the issues are:

- plant, i.e. the process to be controlled
- objectives
- sensors
- actuators
- communications
- computing
- algorithms
- architectures and interfacing
- accounting for disturbances and uncertainties ...”
Skogestad and Postlewaite, 96: “The process of designing a control system makes many demands of the engineering team. The steps to be followed are:

- Plant study and modelling
- Determination of sensors and actuators (measured and controlled outputs, control inputs)
- Performance specifications
- Control design
- Simulation tests
- Implementation ....”
THE CONTROL SYSTEM DESIGN

Steps:

Identification and/or Modelling:
1. Formulate a nonlinear state-space model based on physical knowledge.
2. Determine the steady-state operating point about which to linearize.
3. Introduce deviation variables and linearize the model.

needs good system knowledge
needs criteria choice

Us of various methods:
- Internal model control, Pole placement
- Predictive control, LQ control
- Hinf control .....
objective of a control system:
make the system output behave in a desired way by manipulating the plant input.

The regulator problem is to reject (or reduce) the effect of some disturbance or measurement noise on the output $y$.

The servo problem is to keep the output close to a given reference input $r$. 

In practice the feedback control loop is of the form:
FEEDBACK STRUCTURE

In the following:
only SISO (Single Input Single Output) systems
are considered, for sake of simplicity.
Most of results can be extended to MIMO systems
(Multi Input Multi Output)

SOME CONTROL STRUCTURES

« Classical » one degree-of-freedom structure

Two degree-of-freedom structure

RST structure
Classical one degree-of-freedom structure

**PLANT = G(s)**

**CONTROLLER = K(s)**

**Feedback Structure**
FEEDBACK STRUCTURE

Two degree-of-freedom structure

\[ r(t) \rightarrow K_p(s) \rightarrow d_i(t) \rightarrow G(s) \rightarrow d_y(t) \rightarrow y(t) \rightarrow n(t) \]

FEEDFORWARD

Improves tracking performance

FEEDBACK
Objectives of any control system:

shape the response of the system to a given reference and get (or keep) a stable system in closed-loop, with desired performances, while minimising the effects of disturbances and measurement noises, and avoiding actuators saturation, this despite of modelling uncertainties, parameter changes or change of operating point.
Objectives of any control system

- **Nominal stability (NS):** The system is stable with the nominal model (no model uncertainty)

- **Nominal Performance (NP):** The system satisfies the performance specifications with the nominal model (no model uncertainty)

- **Robust stability (RS):** The system is stable for all perturbed plants about the nominal model, up to the worst-case model uncertainty (including the real plant)

- **Robust performance (RP):** The system satisfies the performance specifications for all perturbed plants about the nominal model, up to the worst-case model uncertainty (including the real plant)
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PERFORMANCE ANALYSIS

Time domain performances
Classical performance indices

Rise time: the time, usually required to be small, that it takes for the output to first reach 90% of its final value.

Settling time: the time after which the output remains within 5% of its final value, which is also usually required to be small.

Overshoot: the peak value divided by the final value: should typically be 1.2 (20%) or less.

Decay ratio: the ratio between the second and first peaks, which should typically be 0.3 or less.

Steady-state offset: the difference between the final value and the desired final value, this offset is usually required to be small.
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Time domain performances

- Initial slope: \( \frac{1}{\text{time constant}} = a \)
- 63% of final value at \( t = \) one time constant

Rise time: \( T_r \)
Settling time: \( T_s \)
Peak value

\[ c(t) \]

\( c_{\text{max}} \)
\( 1.02c_{\text{final}} \)
\( c_{\text{final}} \)
\( 0.98c_{\text{final}} \)
\( 0.9c_{\text{final}} \)
\( 0.1c_{\text{final}} \)
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Decay ratio

Exponential decay generated by real part of complex pole pair

Steady state errors

Sinusoidal oscillation generated by imaginary part of complex pole pair
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Time domain performances: other criteria

ISE (Integral Square Error) \[ J_{ISE} = \int_{0}^{\infty} |e(t)|^2 dt \ ; \ e = r - y \]

ITAE (Integral Time weighted Absolute Error) \[ J_{ITAE} = \int_{0}^{\infty} t|e(t)| dt \ ; \ e = r - y \]

A better and more advisable index should include the control input effect

\[ J_{eu} = \sqrt{\int_{0}^{\infty} \left( Q|e(t)|^2 + R|u(t)|^2 \right) dt} \]
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Frequency performance criteria

PLANT = G(s)
CONTROLLER = K(s)
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Frequency domain performances: Stability

The closed-loop system is internally stable iff all the transfer functions previously defined are stable « roots of the Characteristic polynomial »
This requires closed-loop stability

1. The poles of the closed-loop system are evaluated. The system is stable if and only if all the closed-loop poles are in the open left-half plane (LHP) (that is, poles on the imaginary axis are considered “unstable”). The poles are also equal to the eigenvalues of the state-space matrix A, and this is usually how the poles are computed numerically.

2. The frequency response (including negative frequencies) of \( L(j\omega) \) is plotted in the complex plane and the number of encirclements it makes of the critical point (-1) is counted. By Nyquist’s stability criterion closed-loop stability is inferred by equating the number of encirclements to the number of open-loop unstable poles (RHP-poles).

\[
\left| L(j\omega_{180}) \right| < 1; \quad \angle L(j\omega_{180}) = -180^\circ
\]
Firstly, SISO case

The output & the control input satisfy the following equations:

\[
\begin{align*}
    y(s) &= \frac{1}{1 + G(s)K(s)} (GKr + d_y - GKn + Gd_i) \\
    u(s) &= \frac{1}{1 + K(s)G(s)} (Kr - Kd_y - Kn - KGd_i)
\end{align*}
\]
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Frequency domain performances: Sensitivity functions

\[
y(s) = \frac{1}{1 + G(s)K(s)} (GKr + Gd_i + d_y - GKn)
\]

\[
u(s) = \frac{1}{1 + K(s)G(s)} (Kr - KGd_i - Kd_y - Kn)
\]

Let us define the well known sensitivity functions:

Sensitivity

\[
S(s) = \frac{1}{1 + G(s)K(s)}
\]

Complementary Sensitivity

\[
T(s) = \frac{G(s)K(s)}{1 + G(s)K(s)}
\]

Loop transfer function

\[
L(s) = K(s)G(s)
\]
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Frequency domain performances: criteria

GAIN, PHASE, DELAY and MODULE MARGINS

The *gain margin* indicates the additional gain that would take the closed loop to the critical stability condition

The *phase margin* quantifies the pure phase delay that should be added to achieve the same critical stability condition

The *delay margin* quantifies the maximal delay that should be added in the loop to achieve the same critical stability condition

The *module margin* quantifies the minimal distance between the curve and the critical point (-1,0j): this is a robustness margin
Frequency domain performances: criteria

**GAIN MARGIN**

\[ G_M = -20 \log |L(j \omega_{G_M})|, \quad \text{Arg} L(j \omega_{G_M}) = -\pi \]

Good value for \[ G_m \] : >6dB

Gain difference before instability

Phase difference before instability

Gain margin = \[ G_M = 20 \log a \]

Phase margin = \[ \Phi_M = \alpha \]
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Frequency domain performances: criteria

**PHASE MARGIN**

\[ \Phi_M = \pi + \varphi_0, \quad \varphi_0 = \text{Arg}L(j \omega \Phi_M), \quad |L(j \omega \Phi_M)| = 1 \]

Good value for \( \Phi_m \): > 30\(^\circ\)/40\(^\circ\)
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Frequency domain performances: criteria

DELAY MARGIN: acceptable pure time-delay before instability

\[ \Delta \tau = \frac{\Phi_M}{\omega \Phi_M} \]

Diagram showing the relationship between phase margin and frequency for both systems with and without delay.

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MODULE MARGIN/
Maximum Peak criteria

\[ \Delta M = \min_{\omega} |1 + GK(j\omega)| \]

Also:

\[ \Delta M = \frac{1}{M_S} \]

\[ M_S = \max_{\omega} |S(j\omega)| = \|S\|_{\infty} \]

Good value \( M_S < 2 \) (6 dB)

Gain margin = \( G_M = 20 \log a \)

Phase margin = \( \Phi_M = \alpha \)
**Advantage:** good module margin implies good gain and phase margins

\[ GM \geq \frac{M_S}{M_S - 1} \quad \text{and} \quad PM \geq \frac{1}{M_S} \]

For \( M_S = 2 \), then \( GM > 2 \) and \( PM > 30^\circ \)

Last one : \( M_T = \max_{\omega} |T(j\omega)| \)

Good value \( M_T < 1.5 \) (3.5 dB)
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The MODULE MARGIN is a robustness margin.

\[ \Delta M = \frac{1}{M_S} \]

Indeed, the sensitivity function allows to qualify the robustness of the control system, as

**Closed-loop transfer function**

\[ T_{BF} = \frac{K(s)G(s)}{1 + K(s)G(s)} \]

**Influence of plant modelling errors on the CL transfer function**

\[ \frac{\Delta T_{BF}}{T_{BF}} = \frac{1}{1 + K(s)G(s)} \frac{\Delta G}{G} \]

*Sensitivity function*
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Bandwidth

The concept of bandwidth is very important in understanding the benefits and trade-offs involved when applying feedback control. Above we considered peaks of closed-loop transfer functions, which are related to the quality of the response. However, for performance we must also consider the speed of the response, and this leads to considering the bandwidth frequency of the system.

In general, a large bandwidth corresponds to a faster rise time, since high frequency signals are more easily passed on to the outputs. A high bandwidth also indicates a system which is sensitive to noise and to parameter variations. Conversely, if the bandwidth is small, the time response will generally be slow, and the system will usually be more robust.
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Bandwidth:

Loosely speaking, *bandwidth* may be defined as the frequency range \([w_1, w_2]\) over which control is effective. In most cases we require tight control at steady-state so \(w_1=0\), and we then simply call \(w_2\) the bandwidth. The word “effective” may be interpreted in different ways: globally it means *benefit* in terms of performance.

**Definition 1:** The (closed-loop) bandwidth, \(w_S\), is the frequency where \(|S(jw)|\) crosses –3dB \((1/\sqrt{2})\) from below.

**Remark:** \(|S|<0.707\), frequency zone, where \(e/r = -S\) is reasonably small.
Another interpretation: when it changes the output response. As \( y = Tr \), \( T \) must be sufficiently large.

**Definition 2:** The bandwidth (in term of \( T \)), \( w_T \), is the frequency where \( |T(jw)| \) crosses –3dB \( (1/\sqrt{2}) \) from above.

Remark: In most cases, the two definitions in terms of \( S \) and \( T \) yield similar values for the bandwidth. In other cases, the situation is generally as follows. Up to the frequency \( w_S \), \( |S| \) is less than 0.7, and control is effective in terms of improving performance. In the frequency range \([w_S, w_T]\) control still affects the response, but does not improve performance. Finally, at frequencies higher than \( w_T \), we have \( S \approx 1 \) and control has no significant effect on the response.
The gain crossover frequency:

Definition 3: The bandwidth (crossover frequency), \( w_C \), is the frequency where \( |L(jw)| \) crosses 1 (0dB), for the first time, from above.

Remark: It is easy to compute and usually gives

\[ w_S < w_C < w_T \]

Note that the rise time can often be evaluated as:

\[ t_r = \frac{2.3}{\omega_T} \]
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PERFORMANCE ANALYSIS

Input and Output Performance analysis using the Sensitivity functions

The output & the control input performances can be studied through 4 « sensitivity » functions only.
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\[
y(s) = \frac{1}{1 + G(s)K(s)} (GKr + d_y - GKn + Gd_i)
\]

\[
u(s) = \frac{1}{1 + K(s)G(s)} (Kr - Kd_y - Kn - KGd_i)
\]

Input performance

Output performance
PERFORMANCE ANALYSIS

\[ u(s) = \frac{1}{1 + K(s)G(s)} (Kr - KGd_i - Kd_y - Kn) \]

The effect of the input disturbance \( d_i(t) \) on the plant input \( u(t) + d_i(t) \) (actuator) can be made « small » by making the sensitivity function \( S(s) \) small.

The transfer function \( KS(s) \) should be upper bounded so that \( u(t) \) does not reach the physical constraints, even for a large reference \( r(t) \).

The effect of the measurement noise \( n(t) \) on the plant input \( u(t) \) can be made « small » by making the sensitivity function \( KS(s) \) small (in High Frequencies).

Input performance
PERFORMANCE ANALYSIS

\[
y(s) = \frac{1}{1 + G(s)K(s)} (GKr + Gd_i + d_y - GKn)
\]

The plant output \( y(t) \) can track the reference \( r(t) \) by making the complementary sensitivity function \( T(s) \) equal to 1. (servo pb)

The effect of the output disturbance \( d_y(t) \) (resp. input disturbance \( d_i(t) \)) on the plant output \( y(t) \) can be made « small » by making the sensitivity function \( S(s) \) (resp. \( SG(s) \)) « small »

The effect of the measurement noise \( n(t) \) on the plant output \( y(t) \) can be made « small » by making the complementary sensitivity function \( T(s) \) « small »

\( S(s) + T(s) = 1 \)

Some **trade-offs** are to be looked for
PERFORMANCE ANALYSIS

These trade-offs can be reached if one aims:
- to reject the disturbance effects in low frequency
- to minimize the noise effects in high frequency

We will require:
- S and SG to be small in low frequencies to reduce the load (output and input) disturbance effects on the controlled output
- T and KS to be small in high frequencies to reduce the effects of measurement noises on the controlled output and on the control input (actuator efforts)
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Position control of a DC motor, using internal speed feedback

\[ V_e \rightarrow + \rightarrow A \rightarrow 1/k_v \rightarrow 1/(\tau p + 1) \rightarrow 1/(np) \rightarrow U_0/(2\Pi) \rightarrow V_s \]

\[ V_e \rightarrow - \rightarrow RC_p/(RC_p + 1) \rightarrow 1/k_v \rightarrow 1/k_v \rightarrow \Omega \rightarrow X \]
PERFORMANCE ANALYSIS

Good disturbance rejection

\[ W_B = 15.3 \text{ rad/s} \]

Good noise rejection

\[ W_{BT} = 27.1 \text{ rad/s} \]

Bad: control input sensitive to noise

Bad: Input disturbance \((d_i)\) are not rejected
PERFORMANCE ANALYSIS

Bandwidth:
\[ W_S = 15.3 \text{ rad/s} \]
\[ W_T = 27.1 \text{ rad/s} \]
\[ W_C = 21 \text{ rad/s} \]

And it holds:
\[ W_S < W_C < W_T \]

\[ \omega_C = 27 \text{ rad/s} \]

Phase margin = 72.4 deg
Gain margin = \text{inf}
Module margin < 1.5 db, \( M_T = 0.5 \text{db} \)

Bode Diagrams

Nichols Charts

From: \( U(1) \)
To: \( Y(1) \)
PERFORMANCE ANALYSIS

Bandwidth:

\( W_S = 15.3 \text{ rad/s} \)

\( W_T = 27.1 \text{ rad/s} \)

\( W_c = 21 \text{ rad/s} \)

It holds:

\( W_S < W_c < W_T \)

and:

\[ t_r = \frac{2.3}{\omega_T} = 85 \text{ ms} \]
EXAMPLE using MATLAB

% Determination of the sensitivity functions
G % plant model LTI model
K % controller LTI model
L=series(G,K) % Loop transfer function L=GK
S=inv(1+L); % S = 1/(1+L)
poleS=pole(S)
T= feedback(L,1)
poleT=pole(T)

SG=S*G;;
poleSG=pole(SG)
KS=K*S;
poleKS=pole(KS)

w=logspace(-2,2,500);
subplot(2,2,1), sigma(S,w), title('Sensitivity function')
subplot(2,2,2), sigma(T,w), title('Complementary sensitivity function')
subplot(2,2,3), sigma(SG,w), title('Sensitivity*Plant')
subplot(2,2,4), sigma(KS,w), title('Controller*Sensitivity')
EXAMPLE

- Sensitivity function $S$
- Complementary sensitivity function $T$
- Sensitivity function $SG$
- Sensitivity function $KS$

$S$ and $T$ are the singular values in dB.

$SG$ and $KS$ are also shown for comparison.
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A control system is robust if it is insensitive to differences between the actual system and the model of the system which was used to design the controller.

How to take into account the difference between the actual system and the model?

A solution: using a model set. BUT: very large problem and not exact yet.

A method: these differences are referred as model uncertainty.

The approach:

- determine the uncertainty set: mathematical representation
- check Robust Stability
- check Robust Performance

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\[ P_\Delta(s) = P(s) + w(s)\Delta(s), \quad \sigma[\Delta(j\omega)] < 1, \quad \forall \omega \]
\[ P_\Delta(s) = (I + w(s)\Delta(s))P(s). \]
Let us consider the case:

The loop transfer function is then:

\[ L_p = G_p K = GK(I + w_t \Delta_t) = L + w_t L \Delta_t; \]

Therefore RS \( \iff \) System stable \( \forall L_p \iff L_p \) should not encircle the point -1

\[ RS \iff |w_t L| < |1 + L|, \forall \omega \]

\[ \iff \left| \frac{w_t L}{1 + L} \right| < 1, \forall \omega \]

\[ \iff |w_t T| < 1 \forall \omega \]