Is a mixed design of observer-controllers for time-delay systems interesting?

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Abstract

This paper deals with $H_\infty$ observer-based feedback control for linear time-delay systems, in the framework of delay independent stability. We will propose a new LMI solution to observer-controller design that ensures a disturbance attenuation level for the controlled output as well as for the state estimation error, which is an open problem. This will be compared with a well-known solution and with a usual strategy in control which consists in designing the observer and the controller separately. Our aim is to try to bring a positive answer to the following question: is there an interest to solve the problem in a single (unique) formulation or should we design separately the observer and the controller? An application to a wind tunnel model is provided to emphasize the interest of the given results, particularly in comparison with existing results on $H_\infty$ observer-based control.

Keyword: Time-delay, $H_\infty$, Observer-based controller, LMI

1 Introduction

Many works have been done on the study and control of time-delay systems during the last decades [1, 2], in particular on the stability analysis, either in the delay-independent or in the delay-dependant frameworks (sometimes with robustness constraints) [3, 4]. While the stabilization problem has received a considerable attention [5, 6], the observer design problem was few studied [7, 8, 9, 10]. For sake of simplicity, we consider single delayed-state systems as:

$$
\begin{align*}
\dot{x} &= A_0 x + A_1 x_h + B u + E w \\
y &= C x + F w \\
z &= D x \\
x(\theta) &= \phi(\theta); \quad \theta \in [-h,0] 
\end{align*}
$$

(1)

where $x \in \mathbb{R}^n$ is the state vector, $x_h = x(t-h)$, $u \in \mathbb{R}^r$ is the control input, $y \in \mathbb{R}^p$ is the output measurement vector, $z \in \mathbb{R}^m$ is the controlled output, $w \in \mathbb{R}^q$ is the square-integrable disturbance vector, $\phi(\theta) \in C[-h,0]$ is the functional initial condition and $h \in \mathbb{R}^+$ is the delay.

The aim of this paper is to tackle the $H_\infty$ observer-based controller design problem of time-delay systems [11, 12] (which are of great importance when the estimation error may also be used for instance for diagnosis purpose), and to bring an answer to the following question: is there an interest to solve the problem in a single (unique) formulation or should we design separately the observer and the controller? This problem is here studied in the delay independent case and the comparison of different methods relies on the performance of the methods w.r.t the disturbance attenuation ($H_\infty$ criterion).

We will propose a new $H_\infty$ robust observer-controller in an LMI framework. To give the reader a guided method of observer-controller design, the provided results will be compared with existing strategies (e.g [11]) and with a separate design of the $H_\infty$ observer [13] and the $H_\infty$ controller [5]. On the other hand, as the delay value is needed in the observer equation, a robustness analysis is performed in the case where the delay used in the observer equation is different from the system delay (due to uncertainties).

The contributions of the paper are the following:
We will provide an LMI solution to the $H_\infty$ observer-controller design, much simpler and powerful that previous results.

The given solution allows to get an $H_\infty$ disturbance attenuation property for the controlled output and for the state estimation errors as well, while, in previous contribution, only the controlled output is considered which makes the observer useless.

In practice, the observer delay may be different from the system one (which may be unknown or difficult to measure). In this case, we provide here a robustness analysis (with respect to delay uncertainties) and give an evaluation of the maximum allowable uncertainty that preserves the stability of the (extended) closed-loop system.

The outline of this paper is as follows. In section 2, the result of [11] is recalled. Section 3 is concerned with the $H_\infty$ dynamic output feedback controller obtained by a separate design of the observer and the controller. In section 4 a new simple method of $H_\infty$ observer-controller design is proposed in an LMI framework. The robustness of observer-controller scheme is studied in section 5 to get the maximal delay uncertainty that ensures stability. The illustrative example, i.e., the wind tunnel model, is presented in section 6. Finally, some concluding remarks end the paper.

2 Background

The result of [11] concerns observer-based controllers of the form (2). Note that it includes a specific term $EG\hat{x}(t)$ that represents the coupling within the observer and the control. In an $H_\infty$ framework it represents an estimation of the worst possible disturbance.

$$
\begin{align*}
\hat{x}(t) &= (A_0 + EG)\hat{x}(t) + A_1\hat{x}(t-h) + Bu(t) - L(C\hat{x}(t) - y) \\
u(t) &= K\hat{x}(t)
\end{align*}
$$

(2)

The observer and the control are then obtained through two coupled Riccati equations including 7 parameters (2 matrices and 5 constants) which is very involved to use. Nevertheless the estimation error stability is guaranteed for closed-loop systems only and no performance (in terms of $H_\infty$ gain) is ensured for the observer. On the other hand, such a performance is obtained for the closed-loop system, as stated below.

**Proposition 1** Consider the time-delay system (1) (with $F = I_p$) and the observer-based controller (2), and suppose that the control parameters are given by:

$$
K = -\frac{1}{\epsilon_o}BTP_c, \quad G = \frac{1}{\gamma^2\epsilon_o}ETP_c, \quad L = \frac{1}{\epsilon_o}P_oC^T
$$

where $\epsilon_o$ and $\epsilon_c$ are some positive constants, and $P_c$ and $P_o$ are positive definite solution matrices to the following Riccati-like equations for some for some positive constants $\delta_c$ and $\delta_o$ and some positive-definite weighting matrices $Q_c$ and $Q_o$:

$$
\begin{align*}
A_c^TP_c + P_cA_0 - \frac{1}{\epsilon_c}P_c(BBT - \frac{1}{\delta_c}A_1A_1^T - \frac{1}{\gamma^2\epsilon_c}EE^T)P_c + \epsilon_c(\delta_cI_n + D^TD + Q_c) &= 0 \\
(A_0 + EG)P_o - \frac{1}{\epsilon_o}P_o(C^TC - \frac{1}{\delta_o}K^TK - \frac{1}{\gamma^2\epsilon_o}I_o)P_o + P_o(A_0 + EG)^T + \epsilon_o(\delta_oA_1A_1^T + EE^T + Q_o) &= 0
\end{align*}
$$

Then, for all $h$, the closed-loop system is asymptotically stable and such that $\|T_{zw}\|_\infty \leq \gamma$.

3 A preliminary approach to $H_\infty$ dynamic measurement feedback

In this part the proposed method is to design an observer independently of the control. We assume that the observer has been designed using the results of [13] while the control has been obtained using [5]. We then consider an $H_\infty$ observer-based control using Luenberger-type observer as:

$$
\begin{align*}
\hat{x}(t) &= A_0\hat{x}(t) + A_1\hat{x}(t-h) + Bu(t) - L(C\hat{x}(t) - y(t)) \\
u(t) &= K\hat{x}(t)
\end{align*}
$$

(4)
where \( \hat{x}(t) \in \mathbb{R}^n \) is the estimated state of \( x(t) \) and \( L \) (resp. \( K \)) is the \( n \times p \) (resp. \( r \times n \)) constant observer (resp. controller) gain matrix.

Noting \( e(t) := x(t) - \hat{x}(t) \) and \( x_e(t) = [ x(t) \ e(t) ]^T \), the closed-loop system with observer and control (4) is:

\[
\begin{align*}
\dot{x}_e(t) &= \begin{bmatrix} A_0 + BK & -BK \\ 0 & A_0 - LC \end{bmatrix} x_e(t) + \begin{bmatrix} A_1 & 0 \\ 0 & A_1 \end{bmatrix} x_e(t - h) + \begin{bmatrix} E \\ E - LF \end{bmatrix} w(t) \quad (5)
\end{align*}
\]

The characteristic polynomial of the extended system is:

\[
\det(sI_n - A - BK - A_1 e^{-sh}) \times \det(sI_n - A + LC - A_1 e^{-sh})
\]

If the observer and the control are designed separately then the closed-loop system with the dynamic measurement feedback is stable (i.e. the separation principle holds), given that the control and observer systems are stable and the eigenvalues of (5) can be obtained directly from them. We also have that the \( H_\infty \)-norm of the transfer function between the disturbance and the controlled output of (5), \( \tilde{T}_{zw} \), is bounded as follows

\[
\| \tilde{T}_{zw} \|_\infty \leq \| T_{zw} \|_\infty + \| H \|_\infty \| T_{ew} \|_\infty \leq \gamma_c + \gamma_o \| H \|_\infty \quad (6)
\]

where \( T_{zw}(s) = D[sI_n - A_0 - BK - A_1 e^{-sh}]^{-1}E \), \( T_{ew}(s) = [sI_n - A_0 + LC - A_1 e^{-sh}]^{-1}(E - LF) \),

\( H(s) = D[sI_n - A_0 - BK - A_1 e^{-sh}]^{-1}BK \).

**Remark 1** As the design is made in two steps, the obtained disturbance attenuation level for the controlled output cannot be known a priori but only a posteriori by (6). Nevertheless, thanks to the triangular form of the extended system (5), the attenuation bound for the estimation error does not change when the system is in closed-loop.

### 4 A new \( H_\infty \) observer-controller design method

This section contains the main contribution of this paper, i.e. the design of an \( H_\infty \) observer-based controller in an LMI framework. The provided method ensures, not only the stability of the extended system, but also an a priori \( H_\infty \) attenuation property of the disturbance, for the controlled output and the estimation error, which is not the case in the previous mixed design methods. This guarantees that the observer can be used, in this control structure, to get a good estimation of the state variables. For instance, this can be useful also for diagnosis purpose where the residuals, generated as the difference between measured and estimated variables, are used for fault detection. First, an \( H_\infty \) stability criterion is recalled for a time-delay system of the form (1), as an extension of results in [1], or in a similar form as in [14].

**Lemma 1** Consider system (1). Given a positive scalar \( \gamma \), if there exist some positive definite matrices \( P = P^T \) and \( S \) such that

\[
\mathcal{L} = \begin{bmatrix}
A_0'P + PA_0 + D'D + S & PA_1 & PE \\
A_1'P & -S & 0 \\
E'P & 0 & -\gamma^2I_q
\end{bmatrix} < 0,
\]

then the trivial solution of (1) with \( w \equiv 0, u \equiv 0 \), is asymptotically stable for any delay, and \( \| T_{zw}(s) \|_\infty \leq \gamma \), for zero initial condition and some positive scalar \( \gamma \).

Note that this result can directly be extended to the case of time-varying delay, as precised in [1]. Finally the optimal value of the attenuation bound \( \gamma \) can be found by solving

\[
\gamma_{\min}^2 = \min_{P, S} \gamma^2
\]

s.t. \( \mathcal{L} < 0, \ P > 0, \ S > 0, \)

3
The extended closed-loop system with observer and control (8) is:

\[
\begin{align*}
\dot{x} &= A_0x + A_1\dot{x} + Bu - L(C\dot{x} - y) + G\dot{x} \\
u &= K\dot{x}
\end{align*}
\] (8)

The extended closed-loop system with observer and control (8) is:

\[
\begin{align*}
\dot{\hat{x}} &= \begin{bmatrix} A_0 + BK & -BK \\ -G & A_0 - LC + G \end{bmatrix} \hat{x} + \begin{bmatrix} A_1 & 0 \\ 0 & A_1 \end{bmatrix} (x_e) + \begin{bmatrix} E \\ E - LF \end{bmatrix} w \\
\hat{z} &= A_0\hat{x} + A_1(x_e) + \mathcal{E}w
\end{align*}
\] (9)

The aim is here to provide results that ensure $H_\infty$ stability for the controlled output $z$ and the estimation error $e$. Two different cases are investigated:

1. obtain an $H_\infty$ stabilization of a new controlled output, combining $z$ and $e$ (with a unique attenuation bound $\gamma$)

2. get the $H_\infty$ stabilization of the controlled output $z = Dx$ and of the estimation error $e$ (with two different attenuation bounds, $\gamma_c$ and $\gamma_o$ resp.).

**Definition 1** The system (8) is said to be an $H_\infty$ observer-controller for system (1) if

- the trivial solution of (9) with $w(t) \equiv 0$, is asymptotically stable, and
- 1. $\|T_{z,w}(s)\|_\infty \leq \gamma$ for zero initial condition and some positive scalar $\gamma$, with $z_e = \begin{bmatrix} D & 0 \end{bmatrix} x_e = Dz_e$, or
- 2. $\|T_{z,w}(s)\|_\infty \leq \gamma_c$ and $\|T_{z,w}(s)\|_\infty \leq \gamma_o$, with $z_e = \begin{bmatrix} D & 0 \end{bmatrix} x_e$ and $z_o = \begin{bmatrix} 0 & I_n \end{bmatrix} x_e$, for zero initial condition and some positive scalars $\gamma_c$ and $\gamma_o$.

4.1 Case 1

This first case consists in applying the result of lemma 1 to the extended system (9). This leads to the following solution.

**Theorem 1** Consider the time-delay system (1) and the observer-controller (8). Given a positive scalar $\gamma$, if there exist positive definite matrices $P_c = P_c^T$, $P_o = P_o^T$, $S_c$ and $S_o$, and some matrices $X \in \mathbb{R}^{m \times n}$, $Y \in \mathbb{R}^{n \times p}$ satisfying the following matrix inequality:

\[
\mathcal{L}_{oc} = \begin{bmatrix} M_c & 0 & A_1P_c & 0 & E & P_cD^T \\
* & M_o & 0 & P_oA_1 & P_oE - LF & 0 \\
* & * & -S_c & 0 & 0 & 0 \\
* & * & * & -S_o & 0 & 0 \\
* & * & * & * & -\gamma^2I_q & 0 \\
* & * & * & * & * & -I_n \end{bmatrix} < 0, \tag{10}
\]

where $*$ means the symmetric element, and $M_c = A_0P_c + P_cA_0^T + BX + X^T B^T + S_c$, $M_o = A_0^TP_o + P_oA_0 - P_c^{-1}(BX + X^T B^T)P_c^{-1} + C^TY + YC + I_n + S_o$, then the system (8) is an $H_\infty$ observer-controller according to Definition 1, with the disturbance attenuation level $\gamma$ and the observer-controller gains:

\[
L = -P_o^{-1}Y, \quad K = XP_c^{-1}, \quad G = -P_o^{-1}K^TB^TP_c^{-1}
\]

**Proof:** The aim of this paper is to apply Lemma 1 to the extended system, which leads to the LMI:

\[
\begin{bmatrix} \mathcal{A}_d^TP + P_\mathcal{A}_0 + D^T \mathcal{D} + S & P_\mathcal{A}_1 & \mathcal{P}E \\
\mathcal{A}_d^TP & -S & 0 \\
\mathcal{E}' \mathcal{P} & 0 & -\gamma^2I_q \end{bmatrix} < 0 \tag{11}
\]
Assuming that $\mathcal{P}$ and $\mathcal{S}$ are of the form:

$$\mathcal{P} = \text{diag}(P_1, P_2) \quad \mathcal{S} = \text{diag}(S_1, S_2)$$ (12)

we obtain:

$$\begin{bmatrix} L_{11} & -P_1BK - C^TP_2 & P_1A_1 & 0 & P_1E \\ * & L_{22} & 0 & P_2A_1 & P_2(E - LF) \\ * & * & -S_1 & 0 & 0 \\ * & * & * & -S_2 & 0 \\ * & * & * & * & -\gamma^2 I_q \end{bmatrix} < 0$$

with $L_{11} = (A^T + K^TB^T)P_1 + P_1(A + BK) + D^TD + S_1$, $L_{22} = (A^T - C^TL^T + G^T)P_2 + P_2(A - LC + G) + I_n + S_2$. Now let us choose $G$ such as $G^TP_2 = -P_1BK$ and multiplying both sides of the previous LMI by $\text{diag}([P_1^{-1}, I_n, P_1^{-1}, I_n, I_q])$ it leads to:

$$\mathcal{L}_2 < 0, \text{ with } \mathcal{L}_2 = \begin{bmatrix} L_{211} & 0 & A_1P_1^{-1} & 0 & E \\ * & L_{222} & 0 & P_2A_1 & P_2(E - LF) \\ * & * & -P_1^{-1}S_1P_1^{-1} & 0 & 0 \\ * & * & * & -S_2 & 0 \\ * & * & * & * & -\gamma^2 I_q \end{bmatrix}$$

and $L_{211} = P_1^{-1}(A^T + K^TB^T) + (A + BK)P_1^{-1} + P_1^{-1}D^TDP_1^{-1} + P_1^{-1}S_1P_1^{-1}$, $L_{222} = (A^T - C^TL^T + G^T)P_2 + P_2(A - LC + G) + I_n + S_2$. Noting $P_c = P_1^{-1}$, $P_o = P_2$, $S_c = P_1^{-1}S_1P_1^{-1}$, $S_o = S_2$, $X = KP_1^{-1}$, $Y = -P_2L$, we get:

$$\mathcal{L}_2 = \begin{bmatrix} L_{211} & 0 & A_1P_1 & 0 & E \\ * & L_{222} & 0 & P_2A_1 & P_oE + YF \\ * & * & -S_c & 0 & 0 \\ * & * & * & -S_o & 0 \\ * & * & * & * & -\gamma^2 I_q \end{bmatrix} < 0$$ (13)

with $L_{211} = P_cA^T + AP_c + XT^TB^T + BX + P_cD^TD + Sc_1$, $L_{222} = A^TP_o + P_oA - P_c^{-1}(XT^TB^T + BX)P_c^{-1} + C^TY^T + YC + I_n + S_o$. Using the Schur complement, it leads to the inequality (10).

If the minimal attenuation bound is to be searched, then the following optimization problem has to be solved:

$$\gamma_{\text{min}}^2 = \min_{P_c, P_o, X, Y, S_c, S_o} \gamma^2$$

s.t. $\mathcal{L}_{oc} < 0$, $P_c > 0$, $P_o > 0$, $S_c > 0$, $S_o > 0$, $\gamma^2 > \text{tol}$

(14)

Of course the problem to be solved (14) is not convex due to the term $P_c^{-1}(BX + XT^TB^T)P_c^{-1}$ in $M_o$. Note that, denoting $Z = -G^TP_o = P_c^{-1}BK$, this can be rewritten as:

$$M_o = A^TP_o + P_oA + C^TY^T + YC + I_n + S_o - Z^T - Z$$ (15)

A first attempt to solve this non convex problem is given below.

**Proposition 2** A solution to the observer-controller design is to follow the iterative procedure:

**Step 1:** Initialisation: solve the LMI problem (14) with $M_o$ of the form (15), $Z$ being unknown and get $Z$, $\gamma^i = \gamma^i_{\text{min}}$. Set test $= 1$, tol $= 1e - 3$, $i = 1$ and $N_{\text{iter}} = 50$.

**Step 2:** While (test$=1$) and ($i < N_{\text{iter}}$),

Solve the LMI problem (14) with $M_o$ of the form (15) ($Z$ being the one obtained at step 1) and get $P_c$, $P_o$, $X$, $Y$, $S_c$, $S_o$, $\gamma_{\text{min}}$

Calculate $Z = P_c^{-1}BP_c^{-1}$ and set $\gamma^i = \gamma^i_{\text{min}}$, test$=(\frac{\gamma^i - \gamma^{i-1}}{\gamma^i} > \text{tol})$, $i = i + 1$, $\gamma^{i-1} = \gamma^i$, end.

**Step 3:** If test$=0$, then $\gamma_{\text{min}} = \gamma_1$. calculate $L = -P_o^{-1}Y$, $K = XP_c^{-1}$ and $G = -P_o^{-1}K^TB^TP_c^{-1}$.
Step 4: Check the $H_{\infty}$ closed-loop stability by solving the optimisation problem (7) (following Lemma 1) on the extended system (9) with $L$, $K$ and $G$ obtained at step 3. This allows to get the minimal attenuation bound ensured with the $H_{\infty}$ stabilizing observer-controller.

Note that, in order to reduce the convergence time when the attenuation bound approaches zero, the optimal problem (14) is solved with the constraints $\gamma > 0.01$. This will avoid to do many iterations when $\gamma$ is approaching 0.

4.2 Case 2

In that case the result of lemma 1 is applied to the extended system (9) in both configurations:

1. $z_c = [D \ 0 \ x_c = D_c x_c$ (requiring an attenuation bound $\gamma_c$)

2. $z_o = [0 \ I_n \ x_c = D_o x_c$ (requiring an attenuation bound $\gamma_o$)

This leads to the following result:

**Theorem 2** Consider the time-delay system (1) and the observer-controller (8). Given some positive scalars $\gamma_c$ and $\gamma_o$, if there exist positive definite matrices $P_c = P_c^T$, $P_o = P_o^T$, $S_c$ and $S_o$, and some matrices $X \in \mathbb{R}^{n \times n}$, $Y \in \mathbb{R}^{n \times p}$ satisfying the following matrix inequalities:

$$
\mathcal{L}_c = \begin{bmatrix}
M_c & 0 & A_1 P_c & 0 & E & P_c D^T \\
* & M_o & 0 & P_o A_1 & P_o E + Y F & 0 \\
* & * & -S_c & 0 & 0 & 0 \\
* & * & * & -S_o & 0 & 0 \\
* & * & * & * & -\gamma_c^2 I_q & 0 \\
* & * & * & * & * & -I_n
\end{bmatrix} < 0,
$$

$$
\mathcal{L}_o = \begin{bmatrix}
M_c & 0 & A_1 P_c & 0 & E \\
* & M_o & 0 & P_o A_1 & P_o E + Y F \\
* & * & -S_c & 0 & 0 \\
* & * & * & -S_o & 0 \\
* & * & * & * & -\gamma_o^2 I_q
\end{bmatrix} < 0,
$$

where $*$ means the symmetric element, and

$$
M_c = A_0 P_c + P_c A_0^T + B X + X^T B^T + S_c,
M_o = A^T P_o + P_o A - P_c^{-1} (B X + X^T B^T) P_c^{-1} + C^T Y^T + Y C + S_o,
M_o = A^T P_o + P_o A - P_c^{-1} (B X + X^T B^T) P_c^{-1} + C^T Y^T + Y C + I_n + S_o
$$

then the system (8) is an $H_{\infty}$ observer-controller according to Definition 1, with the disturbance attenuation levels $\gamma_c$, $\gamma_o$ (respectively for the controlled output $z = D x$ and for the state estimation error $e$) and the observer-controller gains:

$$
L = -P_o^{-1} Y, \ K = X P_c^{-1}, \ G = -P_o^{-1} K^T B^T P_c^{-1}
$$

If a minimal attenuation bound is to be searched, a solution is to solve the following optimization problem:

$$
\gamma_{\text{min}} = \min_{P_c, P_o, \gamma_c, \gamma_o, S_c, S_o} \frac{1}{2} (\gamma_c + \gamma_o)
$$

subject to $\mathcal{L}_c < 0$, $\mathcal{L}_o < 0$, $P_c > 0$, $P_o > 0$, $S_c > 0$, $S_o > 0$.

**Proof:** The proof directly follows the methodology of the previous proof of Theorem 1. Applying Lemma 1 for both configurations ($z_c$ and $z_o$), leads to:

with $z_c$ we obtain $L2$ with $L2_{11} = P_c A^T + A P_c + B X + X^T B^T + P_c D^T D P_c + S_c$, $L2_{22} = A^T P_o + P_o A - P_c^{-1} (B X + X^T B^T) P_c^{-1} + C^T Y^T + Y C + S_o$
with $z_o$: We obtain $L2_{11} = P_c A^T + A P_c + B X + X^T B^T + S_c$, $L2_{22} = A^T P_o + P_o A - P_c^{-1}(B X + X^T B^T) P_c^{-1} + C^T Y^T + Y C + S_o + I_n$.

which leads to the above theorem. □

Finally the intuitive procedure given above to overcome the nonlinearity in the matrix inequality can be directly applied to this case, with the new optimisation problem (19).

5 Robustness analysis w.r.t delay uncertainty

In this part we assume that the delay is uncertain, i.e the delay of the real system is $h = d + \theta $ and may be different from the one used in the observer (the nominal delay $d$). Note that the method here used has been developed in [15]. It can be directly extended to other types of uncertainties.

In this case, the observer-controller is then of the form

$$
\begin{align*}
\dot{x} &= A_0 x + A_1 \dot{x}(t - d) + Bu - L(C \dot{x} - y) + G \hat{x} \\
u &= K \hat{x}
\end{align*}
$$

(20)

and the extended closed-loop system is:

$$
\dot{x}_e = 
\begin{bmatrix}
A_0 + BK & -BK \\
-G & A_0 - LC + G
\end{bmatrix}
x_e +
\begin{bmatrix}
A_1 & 0 \\
1 & 0
\end{bmatrix}
(x_e)_h +
\begin{bmatrix}
0 & 0 \\
-A_1 & A_1
\end{bmatrix}
(x_e)_d +
\begin{bmatrix}
E \\
E - LF
\end{bmatrix}
w
$$

(21)

Now, assuming $h = d + \theta$, we can write:

$$
e^{-sh} = e^{-sd} + (e^{-s(d+\theta)} - e^{-sd}) = e^{-sd}(1 - \Delta(s))
$$

with $\Delta(s) = 1 - e^{-sd}$. Therefore the characteristic equation of the above system can be written as:

$$
\Psi(s) = \det[\Psi_0(s)] \det[I_n + \Psi_0^{-1}(s)A_h e^{-sd}\Delta(s)]
$$

where $\Psi_0(s) = s I_{2n} - A_0 - (A_h + A_d) e^{-sd}$. Now, the previous design ensures that the nominal extended system is stable, i.e. $\det[\Psi_0(s)]$ is stable. Then the perturbed closed loop system remains stable if $\det[I_n + \Psi_0^{-1}(s)A_h e^{-sd}\Delta(s)]$ does not change sign when $s$ sweeps the imaginary axis. Invoking Rouché's theorem, it follows that the condition for stability is

$$
\|Q_d(s)\Delta(s)\|_\infty < 1.
$$

(22)

where $Q_d(s) = \Psi_0^{-1}(s)A_h e^{-sd}$. As shown in [15], this means that the maximal uncertainty bound that preserves stability may be determined as:

$$
\theta_{max} = 1/\|s e^{-sd}\Psi_0^{-1}(s)A_h\|_\infty
$$

(23)

Then for all $\theta \in (-\theta_{max}, \theta_{max})$, the determinant has a fixed sign, implying the absence of zero crossings, and henceforth the stability of the perturbed system (provided the nominal one is stable). The result here given can be summarized by the following proposition.

Proposition 3 Let the real system (1) be defined with an uncertain delay $h = d+\theta$, where $d$ is known and $\theta$ is the uncertainty (unknown and bounded). Assume that an observer-controller (8) has been designed in the nominal case (i.e. $\theta = 0$). Then the applied observer-controller in the real case (20) preserves the closed-loop stability for all uncertainty up to

$$
\theta_{max} = 1/\|s e^{-sd}\Psi_0^{-1}(s)A_h\|_\infty
$$

(24)

where $\Psi_0(s) = s I_{2n} - A_0 - (A_h + A_d)e^{-sd}$. 

7
6 Application to a wind tunnel model

This example in [16] is a simplified mathematical model of the Mach number dynamic response to guide vane changes. The delay in one state variable represents the transportation time between the guide vanes of the fan and the test section of the tunnel. It has been tackled also in [17], where an approximation approach is used to design a LQG control, i.e. in the presence of Gaussian noise, and assuming the exact knowledge of the delay. In steady-state operating conditions (fan speed, liquid nitrogen injection rate and gaseous-nitrogen vent rate) the dynamic response of the Mach number is given by the following system [17]:

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix} -0.5091 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -36 & -9.6 \end{bmatrix} x(t) \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} x(t-h) \\
&+ \begin{bmatrix} 0 \\ 0 \\ 36 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} w(t) \\
&:= A_0 x(t) + A_1 x(t-h) + B u(t) + E w(t)
\end{align*}
\]

\[
y(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x(t) + w(t)
\]

and

\[
x(t) = \phi(t); \quad t \in [-h, 0]
\]

where \( h = 0.33 \text{sec.} \), \( x_1 \) is the Mach number, \( x_2 \) is the guide vane angle (the only measurement) and \( x_3 = \dot{x}_2 \). The disturbance is a resistant torque on the input motor.

Note that we have decide to control the Mach number but also the vane angle (to respect the system constraints), i.e.

\[
z(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(t)
\]

Using [8] it can be shown that this system is weakly observable, spectrally observable but is not strongly observable. Also the system is not strongly controllable but is spectrally controllable, and therefore weakly controllable. Note that, in what follows, for simulation purpose an initial value at time \( t = 0 \text{sec.} \) is used to generate a functional initial condition on \( t \in [0, 0.5] \). The observer acts at \( t = 0.5 \text{sec} \). A unit step disturbance is applied at \( t = 10 \text{sec} \).

6.1 Separated design

Now the methodology described in sections 3 is applied to construct an observer independently of the control. The observer-controller is of the form (4). Applying the results in [13] and [5], a solution can be obtained with \( \gamma_o = 2.5 \) and \( \gamma_c = 0.3 \), which gives:

\[
L = \begin{bmatrix} 0 & 0.2048 & 0.000655 \end{bmatrix}^T \quad K = \begin{bmatrix} 0 & -0.14247 & -0.0898 \end{bmatrix}
\]

In this case some fixed output disturbance attenuation level is guaranteed (here \( \|T_{zw}\|_\infty = 0.289 \)) as we can see in figure 1. We can prove that \( \|T_{zw}\|_\infty \) satisfies (see section 3):

\[
\|T_{zw}\|_\infty \leq \|T_{zw}\|_\infty + \|H\|_\infty \|T_{ew}\|_\infty \quad (26)
\]

\[
\leq \gamma_c + \|H\|_\infty \gamma_o \quad (27)
\]

Moreover, in Fig. 2 we can see that the attenuation property of \( T_{ew}(jw) \) is less than the prespecified bound (1.04 instead of 2.5).

The estimated error and controlled output responses are shown in figure 3 for the same \( h = 0.33 \text{sec.} \) and unit step disturbance at \( t = 10 \text{sec} \). This simulation confirms the frequency analysis.

Now, applying the result of section 5 in order to analyse the robustness w.r.t delay uncertainties, we obtain that the maximal delay uncertainty that preserves stability is \( \theta_{max} = 118.7 \text{ sec} \).
6.2 Choi-Chung method

Using the method of [11] we can obtain the following result (note that due to the 7 parameters to be set, this procedure is quite involved and not systematic).

Let us choose the following parameters: $\epsilon_c = 1$, $\epsilon_o = 1$, $\delta_c = 0.01$, $\delta_o = 0.01$ and $Q_o = Q_c = I_2$. For $\gamma = 1.4$ (for the closed-loop system), the ARE (3) gives the corresponding solution:

$$K = \begin{bmatrix} 0 & -0.74116 & -0.80468 \end{bmatrix}^T, G = \begin{bmatrix} 0 & 0.1050 & 0.114 \end{bmatrix}, L = \begin{bmatrix} 0 & 0.36834 & -0.43449 \end{bmatrix}^T$$

On figure 4, it is shown that $\|T_{zw}(j\omega)\|_\infty \leq \gamma$ for all $\omega \in \mathbb{R}$, which corresponds to the disturbance attenuation for $z(t)$. Note that the obtained attenuation level is $\|T_{zw}(j\omega)\|_\infty = 0.438$. This is a few upper to the previous case. The estimated error response $e(t)$ and the controlled output $z(t)$ are shown in figure 6 for $h = 0.33$ sec.

We can note in vector $e(t)$ of figure 6 that, as no robust property is guaranteed for the observer in this case, the disturbance attenuation for the state estimation error may not be good. This can be appreciated in the maximum singular value frequency plot of $T_{ew}(j\omega)$ (Fig. 5).

6.3 Proposed method

In this part the illustration of the new proposed strategy is given. To be complete both cases presented in section 4 are described, i.e when or not different attenuation bounds are required for the observation error and for the controlled output.

6.3.1 Case 1

Following the methodology given in proposition 2, a solution is obtained in 4 iterations and is such that $\gamma_{\text{min}} = 0.01$ and

$$G \simeq 0_{3 \times 3} (\forall i, j, G_{ij} < 10^{-9}), \quad K = \begin{bmatrix} 0 & -4.823 \times 10^6 & -29315 \end{bmatrix}^T, \quad L = \begin{bmatrix} 0 & 0 & 10 \end{bmatrix}$$

Now, when applying the step 4 of proposition 2, we obtain

$\gamma_{\text{step}4} = 1.55 \times 10^{-6}$,

which, as we can see in figures 7 and 8 is a good estimation of the real obtained disturbance attenuation level.

Of course we can note that some gains are very large. This is due to the fact that the minimal attenuation bound is required which, in this case, is near 0. If one aims to solve a sub optimal problem only (i.e. with $\gamma$ given a priori) the gain will be much less large.

With this design the frequency and temporal behaviors of the observer-controller is presented in figures 7, 8 and 9. As we can see on these figures, the designed observer controller scheme is not affected by the disturbance input (neither the controlled output nor the state estimation errors), which is a great advantage compared to both previous designs.

Finally, applying proposition 3, the maximal delay uncertainty that preserves stability is $\theta_{\text{max}} = 118.7$ sec, i.e. the same robustness property can be ensured.
6.3.2 Case 2

Following the same methodology as given in proposition 2, a solution is obtained in 3 iterations and is such that $\gamma_{c}^{\min} = 0.01$ and $\gamma_{o}^{\min} = 0.01$. Also we obtain:

$$G \simeq 0_{3 \times 3} \ (\forall i, j, \ G_{ij} < 10^{-9}), \ K = \begin{bmatrix} 0 \\ -1.5065 \times 10^{5} \\ -9.5832 \end{bmatrix}^T, \ L = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

Now, when applying the step 4 of proposition 2, we obtain

$$\gamma_{\text{step}^4}^{\min} = 1.229 \times 10^{-5},$$

which is a good estimation of the real disturbance attenuation level.

The frequency and time-domain simulations are quite similar to the previous case (and are not plotted here). Of course the simulations show that the disturbance does not affect the closed-loop system. Finally, applying proposition 3, the maximal delay uncertainty that preserves stability is $\theta_{\text{max}} = 118.7 \sec$.

Let us note that, as robustness with large delay uncertainties can be proved in this example for all methodologies, simulations with an observer delay $d$ much different from the system one $h$ could show the robust property. As the transient behavior are very few affected by such a difference, results are not shown here.

To conclude on our method, it clearly points out the efficiency of the LMI formulation, which allows to get a solution even with a great disturbance attenuation property, and with no parameter to be set a priori. Also the $H_{\infty}$ disturbance attenuation can be ensured for both the controlled output and the state estimation errors (with or without different attenuation levels), which greatly improves the results and flexibility of the previous papers.

7 Concluding remarks

Using the results of [13] and [5] we have developed first a delay independent dynamic feedback controlled system with $H_{\infty}$ control as well as $H_{\infty}$ observer performance, by solving two independent Riccati equations with only two parameters for each one. In [11] two coupled Riccati equations have to be solved, including 7 parameters, and $H_{\infty}$ performance is ensured only for the control, not for the observer.

A new observer-controller design is proposed and solved in a procedure including LMIs. While this integrated single design is simple, it allows to get a closed-loop system and an observer which both satisfy an $H_{\infty}$ attenuation property. The proposed method is simple to be solved as it does not contain any parameter to be chosen a priori, which differs from the current solution in the literature. As the solution is based on an optimisation procedure, it is also important to note that one can design (if possible) the best observer-controller w.r.t a disturbance attenuation property but could also design an observer-controller scheme with attenuation levels specified a priori for the estimation errors and for the controlled outputs, allowing to tackle the usual trade-off performance (w.r.t disturbance attenuation) /robustness (w.r.t uncertainties). This emphasizes the great flexibility of the methodology.

The given results could be directly extended to the case of multiple time-delay with uncertainties (on the delay and on the system parameters) and also for time-varying delays. As a further extension, it could be possible to take into account some constraints on the observer and controller gain.

Further study may also concern the derivation of delay-dependent $H_{\infty}$ observer-controller; however, to reduce conservatism, complex Lyapunov-Krasovskii functionals are generally used [1] which may lead in our case to non convex matrix inequalities more difficult to be relaxed to get LMIs.

References


Figure 1: $\sigma_{max}(T_{zw}(jw))$-Separated design

Figure 2: $\sigma_{max}(T_{zw}(jw))$ -Separated design

Figure 3: Controlled output and estimated errors -Separated design
Figure 4: $\sigma_{\max}(T_{zw}(jw))$ - Choi & Chung

Figure 5: $\sigma_{\max}(T_{ew}(jw))$ - Choi & Chung

Figure 6: Controlled output and estimated errors - Choi and Chung
Figure 7: $\sigma_{max}(T_{zw}(jw))$ - New design

Figure 8: $\sigma_{max}(T_{ew}(jw))$ - New design

Figure 9: Controlled output and estimated errors - New design