Robust control of MIMO systems

Olivier Sename

Grenoble INP / GIPSA-lab

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1. Some definitions
   - The $\mathcal{H}_\infty$ norm definition
   - Stability issues
2. Introduction to Linear Matrix Inequalities
   - Background in Optimisation
   - LMI in control
   - Some useful lemmas
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   - $\mathcal{H}_\infty$ norm as a measure of the system gain?
   - How to compute the $\mathcal{H}_\infty$ norm?
   - What is $\mathcal{H}_\infty$ control?
4. Why $\mathcal{H}_\infty$ control is adapted to control engineering?
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8. Introduction to LPV systems and control
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   - LPV observer design
   - Summary of LPV approach interests
About the course

Organization

- 20H lessons in class
- 10H Matlab sessions (E3) and 16H (MISCIT)

Homework

- see given document +
- course complements: tools for norm definitions, and Introduction to Linear Matrix Inequalities
- Exercises
- Matlab project

Evaluation

- One final exam
- Homework
- Ability and report on one Matlab project
Reference books

To be studied during the course

  www.nt.ntnu.no/users/skoge/book, chap 1 to 3 available
  www.ece.lsu.edu/kemin, book slides available
  https://sites.google.com/site/brucefranciscontact/Home/publications,, book available
- Carsten Scherer’s courses
  http://www.dcsc.tudelft.nl/~cscherer/, Lecture slides available (MSc Course "Robust Control", MSc Course "Linear Matrix Inequalities in Control")
- + all the MATLAB demo, examples and documentation on the ’Robust Control toolbox’
  (mathworks.com/products/robust)

Other references (some in french)

  csd.newcastle.edu.au

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Robust control in 1 slide?

- Sensitivity function
  \[ S(s) = \frac{1}{1 + L(s)} \]

- Complementary Sensitivity function:
  \[
  y = \frac{G(s)K(s)}{1 + G(s)K(s)} r \\
  = \frac{L(s)}{1 + L(s)} r = T(s).r
  \]
Robust control in 1 slide?

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  \]

Why \( S \) is the key function in control?

\( S \) allows to characterize many things:

- \( S = 1 - T \longrightarrow S = \frac{r-y}{r} \).
  For performance analysis: \((S(\omega = 0) = \text{steady-state error, bandwidth })\)

- if output disturbance \( d_y \) then, \( \frac{y}{d_y} = S \).
  \( S \) to be minimized!

- distance from -1 to Nyquist plot = \[
  \inf_{\omega} \left| -1 - L(j\omega) \right| = \left[ \sup_{\omega} \left| \frac{1}{1+L(j\omega)} \right| \right]^{-1}
  \]

- Robustness w.r.t model uncertainties

Robust control: Find \( K \) s.t \( S \) satisfies all requirements

O. Sename [GIPSA-lab]
Robust control in 1 slide?

- Sensitivity function \( S(s) = \frac{1}{1+L(s)} \)
- Complementary Sensitivity function:
  \[
  y = \frac{G(s)K(s)}{1 + G(s)K(s)} r = \frac{L(s)}{1 + L(s)} r = T(s).r
  \]

Example: an uncertain mass-spring-damper system controlled by a proportional gain.

Robust stability condition:
\[
|S(j\omega)| < \frac{1}{|W(j\omega)|}, \forall \omega
\]
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Definition of LTI systems

Definition (LTI dynamical system)

Given matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_w}$, $C \in \mathbb{R}^{n_z \times n}$ and $D \in \mathbb{R}^{n_z \times n_w}$, a Linear Time Invariant (LTI) dynamical system ($\Sigma_{LTI}$) can be described as:

$$\Sigma_{LTI} : \begin{cases} \dot{x}(t) = Ax(t) + Bw(t) \\ z(t) = Cx(t) + Dw(t) \end{cases}$$

(1)

where $x(t)$ is the state which takes values in a state space $X \in \mathbb{R}^n$, $w(t)$ is the input taking values in the input space $W \in \mathbb{R}^{n_w}$ and $z(t)$ is the output that belongs to the output space $Z \in \mathbb{R}^{n_z}$.

The LTI system locally describes the real system under consideration and the linearization procedure allows to treat a linear problem instead of a nonlinear one. For this class of problem, many mathematical and control theory tools can be applied like closed loop stability, controllability, observability, performance, robust analysis, etc. for both SISO and MIMO systems. However, the main restriction is that LTI models only describe the system locally, then, compared to nonlinear models, they lack of information and, as a consequence, are incomplete and may not provide global stabilization.
Signal norms

Reader is also invited to refer to the famous book of Zhou et al., 1996, where all the following definitions and additional information are given. All the following definitions are given assuming signals $x(t) \in \mathbb{C}$, then they will involve the conjugate (denoted as $x^*(t)$). When signals are real (i.e. $x(t) \in \mathbb{R}$), $x^*(t) = x^T(t)$.

Definition (Norm and Normed vector space)

- Let $V$ be a finite dimension space. Then $\forall p \geq 1$, the application $\|\cdot\|_p$ is a norm, defined as,

$$\|v\|_p = \left(\sum_i |v_i|^p\right)^{1/p} \quad (2)$$

- Let $V$ be a vector space over $\mathbb{C}$ (or $\mathbb{R}$) and let $\|\cdot\|$ be a norm defined on $V$. Then $V$ is a normed space.
Signal norms

Definition ($\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_\infty$ norms)

- The 1-Norm of a function $x(t)$ is given by,
  \[ \|x(t)\|_1 = \int_0^{+\infty} |x(t)| dt \]  
  (3)

- The 2-Norm (that introduces the energy norm) is given by,
  \[ \|x(t)\|_2 = \sqrt{\int_0^{+\infty} x^*(t)x(t)dt} \]
  \[ = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{+\infty} X^*(j\omega)X(j\omega)d\omega} \]  
  The second equality is obtained by using the Parseval identity.

- The $\infty$-Norm is given by,
  \[ \|x(t)\|_\infty = \sup_t |x(t)| \]  
  \[ \|X\|_\infty = \sup_{Re(s)\geq 0} \|X(s)\| = \sup_\omega \|X(j\omega)\| \]  
  if the signals that admit the Laplace transform, analytic in $Re(s) \geq 0$ (i.e. $\in \mathcal{H}_\infty$).
**L∞ and H∞ spaces**

**Definition (L∞ space)**

$L∞$ is the space of piecewise continuous bounded functions. It is a Banach space of matrix-valued (or scalar-valued) functions on $\mathbb{C}$ and consists of all complex bounded matrix functions $f(j\omega), \forall \omega \in \mathbb{R}$, such that,

$$\sup_{\omega \in \mathbb{R}} \sigma[f(j\omega)] < \infty$$

(7)

**Definition (H∞ and RH∞ spaces)**

$H∞$ is a (closed) subspace in $L∞$ with matrix functions $f(j\omega), \forall \omega \in \mathbb{R}$, analytic in $Re(s) > 0$ (open right-half plane). The real rational subspace of $H∞$ which consists of all proper and real rational stable transfer matrices, is denoted by $RH∞$.

**Example**

In control theory

$$\begin{align*}
\frac{s+1}{(s+10)(s+6)} & \in RH∞ \\
\frac{s+1}{(s-10)(s+6)} & \notin RH∞ \\
\frac{s+1}{(s+10)} & \in RH∞
\end{align*}$$

(8)
\( \mathcal{H}_\infty \) norm

Definition (\( \mathcal{H}_\infty \) norm)

The \( \mathcal{H}_\infty \) norm of a proper LTI system defined by the state space representation \((A, B, C, D)\) from input \( w(t) \) to output \( z(t) \) and which belongs to \( \mathcal{RH}_\infty \), is the induced energy-to-energy gain (induced \( \mathcal{L}_2 \) norm) defined as,

\[
\|G(j\omega)\|_\infty = \sup_{\omega \in \mathbb{R}} \bar{\sigma}(G(j\omega)) = \sup_{w(s) \in \mathcal{H}_2} \frac{\|z(s)\|_2}{\|w(s)\|_2} = \max_{w(t) \in \mathcal{L}_2} \frac{\|z\|_2}{\|w\|_2} \tag{9}
\]

Physical interpretations of the \( \mathcal{H}_\infty \) norm

- This norm represents the maximal gain of the frequency response of the system. It is also called the worst case attenuation level in the sense that it measures the maximum amplification that the system can deliver on the whole frequency set.

- For SISO (resp. MIMO) systems, it represents the maximal peak value on the Bode magnitude (resp. singular value) plot of \( G(j\omega) \), in other words, it is the largest gain if the system is fed by harmonic input signal.

- Unlike \( \mathcal{H}_2 \), the \( \mathcal{H}_\infty \) norm cannot be computed analytically. Only numerical solutions can be obtained (e.g. Bisection algorithm, or LMI resolution).
Well-posedness

Consider

\[ G = -\frac{s - 1}{s + 2}, \quad K = 1 \]

Therefore the control input is non proper:

\[ u = \frac{s + 2}{3} (r - n - d_y) + \frac{s - 1}{3} d_i \]

DEF: A closed-loop system is well-posed if all the transfer functions are proper

\[ \Leftrightarrow \quad I + K(\infty)G(\infty) \text{ is invertible} \]

In the example \( 1 + 1 \times (-1) = 0 \) Note that if \( G \) is strictly proper, this always holds.
Internal stability

DEF: A system is internally stable if all the transfer functions of the closed-loop system are stable.

\[
\begin{bmatrix}
  y \\
  u
\end{bmatrix}
= \begin{bmatrix}
  (I + GK)^{-1}GK & (I + GK)^{-1}G \\
  K(I + GK)^{-1} & -K(I + GK)^{-1}G
\end{bmatrix}
\begin{bmatrix}
  r \\
  d_i
\end{bmatrix}
\]

For instance:

\[
G = \frac{1}{s - 1}, \quad K = \frac{s - 1}{s + 1}, \quad \begin{bmatrix}
  y \\
  u
\end{bmatrix}
= \begin{bmatrix}
  \frac{1}{s + 2} & \frac{s + 1}{(s - 1)(s + 2)} \\
  \frac{s - 1}{s + 2} & -\frac{1}{s + 2}
\end{bmatrix}
\begin{bmatrix}
  r \\
  d_i
\end{bmatrix}
\]

There is one RHP pole (1), which means that this system is not internally stable. This is due here to the pole/zero cancellation (forbidden!!).
Input-Output Stability

Definition (BIBO stability)

A system $G \ (\dot{x} = Ax + Bu; \ y = Cx)$ is BIBO stable if a bounded input $u(.) \ (\|u\|_\infty < \infty)$ maps a bounded output $y(.) \ (\|y\|_\infty < \infty)$.

Now, the quantification (for BIBO stable systems) of the signal amplification (gain) is evaluated as:

$$\gamma_{\text{peak}} = \sup_{0 < \|u\|_\infty < \infty} \frac{\|y\|_\infty}{\|u\|_\infty}$$

and is referred to as the PEAK TO PEAK Gain.

Definition ($L_2$ stability)

A system $G \ (\dot{x} = Ax + Bu; \ y = Cx)$ is $L_2$ stable if $\|u\|_2 < \infty$ implies $\|y\|_2 < \infty$.

Now, the quantification of the signal amplification (gain) is evaluated as:

$$\gamma_{\text{energy}} = \sup_{0 < \|u\|_2 < \infty} \frac{\|y\|_2}{\|u\|_2}$$

and is referred to as the ENERGY Gain, and is such that:

$$\gamma_{\text{energy}} = \sup \omega \|G(j\omega)\| := \|G\|_\infty$$

For a linear system, these stability definitions are equivalent (but not the quantification criteria).
Small Gain theorem

Consider the so called $M - \Delta$ loop.

![Diagram of M - \Delta loop]

**Theorem**

Suppose $M(s)$ in $RH_\infty$ and $\gamma$ a positive scalar. Then the system is well-posed and internally stable for all $\Delta(s)$ in $RH_\infty$ such that $\|\Delta\|_\infty \leq 1/\gamma$ if and only if

$$\|M\|_\infty < \gamma$$
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Introduction to LMIs

Background in Optimisation

Brief on optimisation

Definition (Convex function)

A function $f : \mathbb{R}^m \to \mathbb{R}$ is convex if and only if for all $x, y \in \mathbb{R}^m$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y)$$  \hspace{1cm} (10)

Equivalently, $f$ is convex if and only if its epigraph,

$$\text{epi}(f) = \{(x, \lambda)| f(x) \leq \lambda\}$$  \hspace{1cm} (11)

is convex.

Definition ((Strict) LMI constraint)

A Linear Matrix Inequality constraint on a vector $x \in \mathbb{R}^m$ is defined as,

$$F(x) = F_0 + \sum_{i=1}^{m} F_i x_i \succeq 0 (\succ 0)$$  \hspace{1cm} (12)

where $F_0 = F_0^T$ and $F_i = F_i^T \in \mathbb{R}^{n \times n}$ are given, and symbol $F \succeq 0 (\succ 0)$ means that $F$ is symmetric and positive semi-definite ($\succeq 0$) or positive definite ($\succ 0$), i.e. $\{\forall u | u^T F u(\succ) \geq 0\}$. 
Convex to LMIs

Example

Lyapunov equation. A very famous LMI constraint is the Lyapunov inequality of an autonomous system $\dot{x} = Ax$. Then the stability LMI associated is given by,

$$
\begin{align*}
    x^T P x & > 0 \\
    x^T (A^T P + PA) x & < 0
\end{align*}
$$

which is equivalent to,

$$
F(P) = \begin{bmatrix} -P & 0 \\ 0 & A^T P + PA \end{bmatrix} \prec 0
$$

where $P = P^T$ is the decision variable. Then, the inequality $F(P) \prec 0$ is linear in $P$.

LMI constraints $F(x) \succeq 0$ are convex in $x$, i.e. the set $\{x | F(x) \succeq 0\}$ is convex. Then LMI based optimization falls in the convex optimization. This property is fundamental because it guarantees that the global (or optimal) solution $x^*$ of the minimization problem under LMI constraints can be found efficiently, in a polynomial time (by optimization algorithms like e.g. Ellipsoid, Interior Point methods).
LMI problem

Two kind of problems can be handled

**Feasibility:** The question whether or not there exist elements \( x \in X \) such that \( F(x) < 0 \) is called a feasibility problem. The LMI \( F(x) < 0 \) is called feasible if such \( x \) exists, otherwise it is said to be infeasible.

**Optimization:** Let an objective function \( f : S \rightarrow R \) where \( S = \{ x | F(x) < 0 \} \). The problem to determine

\[
V_{opt} = \inf_{x \in S} f(x)
\]

is called an optimization problem with an LMI constraint. This problem involves the determination of \( V_{opt} \), the calculation of an almost optimal solution \( x \) (i.e., for arbitrary \( \epsilon > 0 \) the calculation of an \( x \in S \) such that \( V_{opt} \leq f(x) \leq V_{opt} + \epsilon \), or the calculation of a optimal solutions \( x_{opt} \) (elements \( x_{opt} \in S \) such that \( V_{opt} = f(x_{opt}) \)).
Semi-Definite Programming (SDP) Problem

LMI programming is a generalization of the Linear Programming (LP) to cone positive semi-definite matrices, which is defined as the set of all symmetric positive semi-definite matrices of particular dimension.

Definition (SDP problem)

A SDP problem is defined as,

\[
\begin{align*}
\min_{x} & \quad c^T x \\
\text{under constraint} & \quad F(x) \succeq 0
\end{align*}
\]  

(15)

where \( F(x) \) is an affine symmetric matrix function of \( x \in \mathbb{R}^m \) (e.g. LMI) and \( c \in \mathbb{R}^m \) is a given real vector, that defines the problem objective.

SDP problems are theoretically tractable and practically:

- They have a polynomial complexity, i.e. there exists an algorithm able to find the global minimum (for a given a priori fixed precision) in a time polynomial in the size of the problem (given by \( m \), the number of variables and \( n \), the size of the LMI).
- SDP can be practically and efficiently solved for LMIs of size up to \( 100 \times 100 \) and \( m \leq 1000 \) see ElGhaoui, 97. Note that today, due to extensive developments in this area, it may be even larger.
The state feedback design problem

Stabilisation

Let us consider a controllable system $\dot{x} = Ax + Bu$. The problem is to find a state feedback $u(t) = -Kx(t)$ s.t the closed-loop system is stable.

Using the Lyapunov theorem, this amounts at finding $P = P^T > 0$ s.t:

$$\begin{align*}
(A - BK)^T P + P (A - BK) &< 0 \\
\iff A^T P + PA - K^T B^T P - PBK &< 0
\end{align*}$$

which is obviously not linear...

Solution; use of change of variables

First, left and right multiplication by $P^{-1}$ leads to

$$P^{-1} A^T + AP^{-1} - P^{-1} K^T B^T - BK P^{-1} < 0$$

$$\iff Q A^T + AQ + Y^T B^T + BY < 0$$

with $Q = P^{-1}$ and $Y = -KP^{-1}$.

The problem to be solved is therefore formulated as an LMI ! and without any conservatism !
The Bounded Real Lemma

The $L_2$-norm of the output $z$ of a system $Σ_{LTI}$ is uniformly bounded by $γ$ times the $L_2$-norm of the input $w$ (initial condition $x(0) = 0$).

A dynamical system $G = (A, B, C, D)$ is internally stable and with an $||G||_∞ < γ$ if and only if there exists a positive definite symmetric matrix $P$ (i.e $P = P^T > 0$) s.t

$$
\begin{bmatrix}
A^T P + P A & P B & C^T D - γ I
\end{bmatrix}
< 0,
\begin{bmatrix}
P
\end{bmatrix} > 0.
$$

(16)

The Bounded Real Lemma (BRL), can also be written as follows (see Scherer)

$$
\begin{bmatrix}
I & 0 & 0
\end{bmatrix}^T
\begin{bmatrix}
0 & P & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
I & 0
\end{bmatrix} > 0
$$

(17)

Note that the BRL is an LMI if the only unknown (decision variables) are $P$ and $γ$ (or $γ^2$).
Quadratic stability

This concept is very useful for the stability analysis of uncertain systems. Let us consider an uncertain system

\[ \dot{x} = A(\delta)x \]

where \( \delta \) is an parameter vector that belongs to an uncertainty set \( \Delta \).

**Theorem**

*The considered system is said to be quadratically stable for all uncertainties \( \delta \in \Delta \) if there exists a (single) "Lyapunov function" \( P = P^T > 0 \) s.t*

\[ A(\delta)^T P + PA(\delta) < 0, \text{ for all } \delta \in \Delta \]  \hspace{1cm} (18)

*This is a sufficient condition for ROBUST Stability which is obtained when \( A(\delta) \) is stable for all \( \delta \in \Delta \).*
Some useful lemmas

- **Schur lemma**: allows to convert a quadratic constraint (ellipsoidal constraint) into an LMI constraint.

Lemma

Let \( Q = Q^T \) and \( R = R^T \) be affine matrices of compatible size, then the condition

\[
\begin{bmatrix}
Q & S \\
S^T & R
\end{bmatrix} \succeq 0
\]

is equivalent to

\[
R \succ 0 \\
Q - SR^{-1}S^T \succeq 0
\]

- **Kalman-Yakubovich-Popov lemma**: convert frequency inequalities into Linear Matrix Inequalities (used in robust control for dissipative systems)

- **Projection Lemma**: allows to eliminate variable by a change of basis (projection in the kernel basis). It is involved in one of the \( H_\infty \) solutions, see (Doyle, Scherer).

- **Completion Lemma**: allows to simplify the number of variables when a matrix and its inverse enter in a LMI.

- **Finsler’s lemma**: This Lemma allows the elimination of matrix variables.
Interests of LMIs

LMIs allow to formulate complex optimization problems into "Linear" ones, allowing the use of convex optimization tools. Usually it requires the use of different transformations, changes of variables ... in order to linearize the considered problems: Congruence, Schur complement, projection lemma, Elimination lemma, S-procedure, Finsler’s lemma ...

Examples of handled criteria

- stability
- $H_\infty$, $H_2$, $H_2/H_\infty$ performances
- robustness analysis: Small gain theorem, Polytopic uncertainties, LFT representations...
- Robust control and/or observer design
- pole placement
- stability, stabilization with input constraints
- Passivity constraints
- Time-delay systems
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How to define the system gain?

**SISO systems**

\( z = Gd \), the gain at a given frequency is simply

\[
\frac{|z(\omega)|}{|d(\omega)|} = \frac{|G(j\omega)d(\omega)|}{|d(\omega)|} = |G(j\omega)|
\]

The gain depends on the frequency, but since the system is linear it is independent of the input magnitude.

**How to generalize to MIMO systems?**

we may select:

\[
\frac{\|z(\omega)\|_2}{\|d(\omega)\|_2} = \frac{\|G(j\omega)d(\omega)\|_2}{\|d(\omega)\|_2} = \|G(j\omega)\|_2?
\]

Which seems to be "independent" of the input magnitude. But this is not a correct definition. Indeed the input direction is of great importance.
The gain of a MIMO system as induced $\mathcal{L}_2$ norm?

Let consider

$$G = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$$

How to define and evaluate its gain?
Consider five different inputs:

| $d_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ | $d_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ | $d_3 = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix}$ | $d_4 = \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix}$ | $d_5 = \begin{pmatrix} 0.6 \\ -0.8 \end{pmatrix}$ |

The input magnitudes are:

$$\|d_1\|_2 = \|d_2\|_2 = \|d_3\|_2 = \|d_4\|_2 = \|d_5\|_2 = 1$$

But the corresponding outputs are

| $z_1 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ | $z_2 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ | $z_3 = \begin{pmatrix} 6.3630 \\ 3.5350 \end{pmatrix}$ | $z_4 = \begin{pmatrix} 0.7070 \\ 0.7070 \end{pmatrix}$ | $z_5 = \begin{pmatrix} -0.2 \\ 0.2 \end{pmatrix}$ |

and the ratio are $\|z\|_2 / \|d\|_2$

| $\|z_1\|_2 / \|d_1\|_2 = 5.83$ | $\|z_2\|_2 / \|d_2\|_2 = 4.47$ | $\|z_3\|_2 / \|d_3\|_2 = 7.27$ | $\|z_4\|_2 / \|d_4\|_2 = 0.99$ | $\|z_5\|_2 / \|d_5\|_2 = 0.28$ |

So the gain value differs function of the input vector direction.
How the Singular Value Decomposition can provide such a MIMO gain definition?

Below is represented $\|z\|_2/\|d\|_2$ as a function of $d_{20}/d_{10}$ (where $d = [d_{10}, d_{20}]^T$)

We can see that, depending on the ratio $d_{20}/d_{10}$, the gain varies between 0.27 and 7.34.

where $\bar{\sigma}(G') = 7.34$ and $\sigma(G') = 0.27$.

We then have these mathematical definitions:

**MAXIMUM SINGULAR VALUE**

$$\max_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \bar{\sigma}(G')$$

**MINIMUM SINGULAR VALUE**

$$\min_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \underline{\sigma}(G')$$
Characterization of the $\mathcal{H}_\infty$ norm as induced $\mathcal{L}_2$ norm

Finally, in the case of a transfer matrix $G(s) : (m \text{ inputs, } p \text{ outputs}) \ u \text{ vector of inputs, } y \text{ vector of outputs.}$

$$\sigma(G(j\omega)) \leq \frac{\|z(\omega)\|_2}{\|d(\omega)\|_2} \leq \bar{\sigma}(G(j\omega))$$

Example of A two-mass/spring/damper system
2 inputs: $F_1$ and $F_2$ 2 outputs: $x_1$ and $x_2$

$G=ss(A,B,C,D)$: LTI system
$[\text{ninf, fpeak}] = \text{hinfnorm}(G)$: Compute $H_\infty$ norm and freq
$\text{norm}(G, \text{inf})$: Compute $H_\infty$ norm
$\text{normhinfd}(G)$: Compute $H_\infty$ norm
$\text{sigma}(G)$: plot max and min SV
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How to compute the $\mathcal{H}_\infty$ norm?

As said before, $\mathcal{H}_\infty$ norm cannot be computed analytically. Only numerical solutions can be obtained (e.g. Bisection algorithm, or LMI resolution).

**Method 1:** Since $\|G(j\omega)\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma(G(j\omega))$, the intuitive computation is to get the peak on the Bode magnitude plot, which can be estimated using a thin grid of frequency points, $\{\omega_1, \ldots, \omega_N\}$, and then:

$$\|G(j\omega)\|_\infty \approx \max_{1 \leq k \leq N} \sigma\{G(j\omega_k)\}$$

**Method 2:** Let the dynamical system $G = (A, B, C, D) \in \mathcal{RH}_\infty$:

$\|G\|_\infty < \gamma$ if and only if $\bar{\sigma}(D) < \gamma$ and the Hamiltonian $H$ has no eigenvalues on the imaginary axis, where

$$H = \begin{pmatrix} A + BR^{-1}D^TC & BR^{-1}B^T \\ -C^T(I_n + DR^{-1}D^T)C & -(A + BR^{-1}D^TC) \end{pmatrix}$$

and $R = \gamma^2 - D^TD$

Use `norm(sys, inf)` or `hinfnorm(sys, tol)` in Matlab.

**Method 3 (Bounded Real Lemma):** A dynamical system $G = (A, B, C, D)$ is internally stable and with an $\|G\|_\infty < \gamma$ if and only is there exists a positive definite symmetric matrix $P$ (i.e $P = P^T > 0$ s.t

$$\begin{bmatrix} A^T & P & A \\ B^T & P & 0 \\ C & -\gamma I & D \\ C^T & DT & -\gamma I \end{bmatrix} < 0, \quad P > 0. \quad (19)$$
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Towards $\mathcal{H}_\infty$ control: the General Control Configuration

This approach has been introduced by Doyle (1983). The formulation makes use of the general control configuration.

![Diagram](image)

$P$ is the generalized plant (contains the plant, the weights, the uncertainties if any); $K$ is the controller. The closed-loop transfer matrix from $w$ to $z$ is given by:

$$T_{zw}(s) = F_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

where $F_l(P, K)$ is referred to as a lower Linear Fractional Transformation.
Problem definition

The overall control objective is to minimize some norm of the transfer function from $w$ to $z$, for example, the $H_\infty$ norm.

Definition ($H_\infty$ optimal control problem)

$H_\infty$ control problem: Find a controller $K(s)$ which based on the information in $y$, generates a control signal $u$ which counteracts the influence of $w$ on $z$, thereby minimizing the closed-loop norm from $w$ to $z$.

Definition ($H_\infty$ suboptimal control problem)

Given $\gamma$ a pre-specified attenuation level, a $H_\infty$ sub-optimal control problem is to design a stabilizing controller that ensures:

$$\|T_{zw}(s)\|_\infty = \max_\omega \bar{\sigma}(T_{zw}(j\omega)) \leq \gamma$$

The optimal problem aims at finding $\gamma_{min}$ (done using $\text{hinfsyn}$ in MATLAB).

Remarks

- It is worth noting that the $H_\infty$ control problem is a disturbance attenuation, formulated in the worst-case performance analysis.
- $z$ is then often defined as an "error signal" (to be minimized)
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Introduction

Objectives of any control system
shape the response of the system to a given reference and get (or keep) a stable system in closed-loop, with desired performances, while minimising the effects of disturbances and measurement noises, and avoiding actuators saturation, this despite of modelling uncertainties, parameter changes or change of operating point.
This is formulated as:

Nominal stability (NS): The system is stable with the nominal model (no model uncertainty)
Nominal Performance (NP): The system satisfies the performance specifications with the nominal model (no model uncertainty)
Robust stability (RS): The system is stable for all perturbed plants about the nominal model, up to the worst-case model uncertainty (including the real plant)
Robust performance (RP): The system satisfies the performance specifications for all perturbed plants about the nominal model, up to the worst-case model uncertainty (including the real plant).
4 Why $\mathcal{H}_\infty$ control is adapted to control engineering?

**The control structure - SISO case**

In the SISO case, it leads to:

$$
\begin{align*}
y &= \frac{1}{1+G(s)K(s)} \left( GK_r + d_y - GK_n + Gd_i \right) \\
u &= \frac{1}{1+K(s)G(s)} \left( Kr - Kd_y - Kn - KGd_i \right)
\end{align*}
$$

**Figure: Complete control scheme**

<table>
<thead>
<tr>
<th>Loop transfer function</th>
<th>$L = G(s)K(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sensitivity function</td>
<td>$S(s) = \frac{1}{1+L(s)}$</td>
</tr>
<tr>
<td>Complementary Sensitivity function</td>
<td>$T(s) = \frac{L(s)}{1+L(s)}$</td>
</tr>
</tbody>
</table>

N.B. $S$ is often referred to as the 'Output Sensitivity'.

O. Sename [GIPSA-lab] 38/161
Classical definitions:

- **Stability**: \( |L(j\omega)\pi| < 1 \), where \( \omega_\pi \) is the phase crossover frequency defined by \( \phi(L(j\omega_\pi)) = -\pi \).
- **Gain Margin**: indicates the additional gain that would take the closed loop to the critical stability condition \( G_M(dB) = -|L(j\omega_\pi)|_{dB} \)
- **Phase margin**: quantifies the pure phase delay that should be added to achieve the same critical stability condition \( \Phi_M = 180^\circ + \text{arg}[L(j\omega_c)], \text{where} \ |L(j\omega_c)| = 1(0dB) \)
- **Delay margin**: quantifies the maximal delay that should be added in the loop to achieve the same critical stability condition, \( PM/\omega_c \)
Robustness margins ...

It is important to consider the module margin that quantifies the minimal distance between the curve and the critical point (-1,0j): this is a robustness margin.

The MODULE MARGIN is a robustness margin. Indeed, the influence of plant modelling errors on the CL transfer function:

\[ T = \frac{K(s)G(s)}{1 + K(s)G(s)} \]

is given by:

\[ \frac{\Delta T}{T} = \frac{1}{1 + K(s)G(s)} \frac{\Delta G}{G} = S \frac{\Delta G}{G} \]

A good module margin implies good gain and phase margins:

\[ GM \geq \frac{M_S}{M_S - 1} \quad \text{and} \quad PM \geq \frac{1}{M_S} \]

For \( M_S = 2 \), then \( GM > 2 \) and \( PM > 30^\circ \)

Last:

\[ MT = \max_\omega |T(j\omega)| \]

A good value: \( MT < 1.5(3.5dB) \)

\[ \Delta M = \min_\omega |1 + GK(j\omega)| \]

\[ M_S = \max_\omega |S(j\omega)| = \|S\|_\infty \]

Good value \( M_S < 2 \) (6dB)
Defining two new ‘sensitivity functions’:

**Plant Sensitivity:** \( SG = S(s).G(s) \) (often referred to as the ‘Input Sensitivity’, e.g in Matlab)

**Controller Sensitivity:** \( KS = K(s).S(s) \)
Using **Matlab**

```matlab
% Determination of the sensitivity functions
L=series(G,Khinf) % Loop transfer function L=GK
S=inv(1+L); % S= 1/(1+L)
poleS=pole(S)
T= feedback(L,1)
poleT=pole(T)
%---------------------------------------------------------------
SG=S*G;
poleSG=pole(SG)
KS=Khinf*S;
poleKS=pole(KS)
%---------------------------------------------------------------
w=logspace(-3,3,500); % to be adjusted
subplot(2,2,1), sigma(S,w), title('Sensitivity function')
subplot(2,2,2), sigma(T,w), title('Complementary sensitivity function')
subplot(2,2,3), sigma(SG,w), title('Sensitivity*Plant')
subplot(2,2,4), sigma(KS,w), title('Controller*Sensitivity')
```

Remark: for SISO systems, use **bodemag** instead of **sigma** but not for MIMO ones!
Performance analysis and specification using the sensitivity functions: the SISO case- Dynamical behavior

As mentioned in Skogestad & Postlethwaite’s book:
The concept of bandwidth is very important in understanding the benefits and trade-offs involved when applying feedback control. Above we considered peaks of closed-loop transfer functions, which are related to the quality of the response. However, for performance we must also consider the speed of the response, and this leads to considering the bandwidth frequency of the system. In general, a large bandwidth corresponds to a faster rise time, since high frequency signals are more easily passed on to the outputs. A high bandwidth also indicates a system which is sensitive to noise and to parameter variations. Conversely, if the bandwidth is small, the time response will generally be slow, and the system will usually be more robust.

Definition

Loosely speaking, bandwidth may be defined as the frequency range $[\omega_1, \omega_2]$ over which control is effective. In most cases we require tight control at steady-state so $\omega_1 = 0$, and we then simply call $\omega_2$ the bandwidth. The word "effective" may be interpreted in different ways: globally it means benefit in terms of performance.
Performance analysis and specification using the sensitivity functions: the SISO case - Bandwidth definitions

Definition ($\omega_S$)

The (closed-loop) bandwidth, $\omega_S$, is the frequency where $|S(j\omega)|$ crosses $-3dB \left(1/\sqrt{2}\right)$ from below.

Remark: $|S| < 0.707$, frequency zone, where $e/r = -S$ is reasonably small

Definition ($\omega_T$)

The bandwidth (in term of $T$), $\omega_T$, is the frequency where $|T(j\omega)|$ crosses $-3dB \left(1/\sqrt{2}\right)$ from above.

Definition ($\omega_c$)

The bandwidth (crossover frequency), $\omega_c$, is the frequency where $|L(j\omega)|$ crosses $1 \left(0dB\right)$, for the first time, from above.
Some remarks

Remark

*Usually* $\omega_S < \omega_c < \omega_T$

Remark

*In most cases, the two definitions in terms of $S$ and $T$ yield similar values for the bandwidth. In other cases, the situation is generally as follows. Up to the frequency $\omega_S$, $|S|$ is less than 0.7, and control is effective in terms of improving performance. In the frequency range $[\omega_S, \omega_T]$ control still affects the response, but does not improve performance. Finally, at frequencies higher than $\omega_T$, we have $S \simeq 1$ and control has no significant effect on the response.*

Remark

*Usually* $\omega_S < \omega_c < \omega_T$

Finally the following relation is very useful to evaluate the rise time:

$$t_r = \frac{2.3}{\omega_T}$$
Performance analysis: answer to ....

Analysis of $S$:
- Steady state error in tracking and output disturbance rejection: $S(\omega = 0) = 0$ ?
- Maximum peak criterion (Module Margin): $\|S\|_{\infty} < 2$ ?
- Bandwidth of $S$

Analysis of $T$:
- Steady state error in tracking: $T(\omega = 0) = 1$ ?.
- Attenuation of measurement noise: $|T(j\omega)|$ small when $\omega \rightarrow \infty$ ?
- Maximum of $T$, $\|T\|_{\infty} < 1.5$ ?
- $\omega_T$, bandwidth of $T$ + rise time evaluation $t_r$

Analysis of $KS$:
- Input saturation: $|u(t)| < |u_{max}|$ ? (where $|u_{max}| < \|KS\|_{\infty}|r_{max}|$).
- Attenuation of measurement noise: $|KS(j\omega)|$ small when $\omega \rightarrow \infty$ ?

Analysis of $SG$:
- Steady state error in input disturbance rejection: $SG(\omega = 0) = 0$ ?
- Attenuation of input disturbance effect: $|SG(j\omega)|$ small in the frequency range of interest ?.
From analysis to specification ... templates

**Objective**: good performance specifications are important to ensure better control system

**Mean**: give some templates on the sensitivity functions

For simplicity, presentation for SISO systems first.

Sketch of the method:

1. Robustness and performances in regulation can be specified by imposing frequential templates on the sensitivity functions.
2. If the sensitivity functions stay within these templates, the control objectives are met.
3. These templates can be used for analysis and/or design. In the latter they are considered as weights on the sensitivity functions.
4. The shapes of typical templates on the sensitivity functions are given in the following slides.

Mathematically, these specifications may be captured by an upper bound, on the magnitude of a sensitivity function, given by another transfer function, as for $S$:

$$|S(j\omega)| \leq \frac{1}{|W_e(j\omega)|}, \quad \forall \omega \Leftrightarrow \|W_eS\|_{\infty} \leq 1$$

where $W_e(s)$ is a **WEIGHT** selected by the designer.
Template on the sensitivity function $S$ - Weighted sensitivity

Typical specifications in terms of $S$ include:

1. Minimum bandwidth frequency $\omega_S$
2. Maximum tracking error at selected frequencies.
3. System type, or the maximum steady-state tracking error $\epsilon_0$
4. Shape of $S$ over selected frequency ranges.
5. Maximum peak magnitude of $S$, $||S||_\infty < M_S$.

The peak specification prevents amplification of noise at high frequencies, and also introduces a margin of robustness; typically we select $M_S = 2$.

How to select the template function

It should:

- be close to the control objectives
- avoid too much under-or over-estimation
- be simple enough to be used later in the control design step
Why $H_\infty$ control is adapted to control engineering?

Template on the sensitivity function $S$

$$S(s) = \frac{1}{1 + K(s)G(s)}$$

$$\frac{1}{W_e(s)} = \frac{s + \omega_b \varepsilon}{s/M_S + \omega_b}$$

Generally $\varepsilon \simeq 0$ is considered, $M_S < 2 (6dB)$ or $(3dB$ - cautious) to ensure sufficient module margin.

$\omega_b$ influences the CL bandwidth: $\omega_b \uparrow$

- faster rejection of the disturbance
- faster CL tracking response
- better robustness w.r.t. parametric uncertainties

$M_S = 2 (6dB)$

$\omega_b$ s.t. $|1/W_e| = 0dB$

$\varepsilon = 1e - 3$

$|S(j\omega)|$
Template on the function $KS$

$$KS(s) = \frac{K(s)}{1 + K(s)G(s)}$$

$$\frac{1}{W_u(s)} = \frac{\varepsilon_1 s + \omega_{bc}}{s + \omega_{bc}/M_u}$$

$M_u$ chosen according to LF behavior of the process (actuator constraints: saturations)

- better limitation of measurement noises
- roll-off starting from $\omega_{bc}$ to reduce modeling errors effects

\[
\begin{array}{c}
\omega_{bc} \text{ for } |1/W_u| = 0 \text{dB} \\
M_u = 2 \\
\varepsilon_c = 1 \times 10^{-3}
\end{array}
\]
Template on the function $T$

$$T(s) = \frac{K(s)G(s)}{1 + K(s)G(s)}$$

$$\frac{1}{W_T(s)} = \frac{\varepsilon_T s + \omega_{bt}}{s + \omega_{bt}/M_T}$$

Generally $\varepsilon_T \simeq 0$ is considered, $M_T < 1.5$ (3dB) to limit the overshoot. $\omega_{bt}$ influences the bandwidth hence the transient behavior of the disturbance rejection properties: $\omega_{bt} \downarrow$

- better noise effects rejection
- better filtering of HF modelling errors
Template on the function $SG$

$$SG(s) = \frac{G(s)}{1 + K(s)G(s)}$$

$$\frac{1}{W_{SG}(s)} = \frac{s + \omega_{SG}\epsilon_{SG}}{s/M_{SG} + \omega_{SG}}$$

$M_{SG}$ allows to limit the overshoot in the response to input disturbances. Generally $\epsilon_{SG} \simeq 0$ is considered, $\omega_{SG}$ influences the CL bandwidth: $\omega_{SG} \uparrow \rightarrow$ faster rejection of the disturbance.
4 Why $\mathcal{H}_\infty$ control is adapted to control engineering?

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Performance analysis and specification using the sensitivity functions: the MIMO case

The output & the control input satisfy the following equations:

\[(I_p + G(s)K(s))y(s) = (GKr + dy - GKn + Gd_i)\]
\[(I_m + K(s)G(s))u(s) = (Kr - Kdy - Kn - Kd_i)\]

BUT : \(K(s)G(s) \neq G(s)K(s)!!\)
Sensitivity functions- The MIMO case

Definitions

Output and Output complementary sensitivity functions:

\[ S_y = (I_p + GK)^{-1}, \quad T_y = (I_p + GK)^{-1}GK, \quad S_y + T_y = I_p \]

Input and Input complementary sensitivity functions:

\[ S_u = (I_m + KG)^{-1}, \quad T_u = KG(I_m + KG)^{-1}, \quad S_u + T_u = I_m \]

Properties

\[ T_y = GK(I_p + GK)^{-1} \]
\[ T_u = (I_m + KG)^{-1}KG \]
\[ S_u K = KS_y \]
Defining two new 'sensitivity functions':

Plant Sensitivity: \( S_y G = S_y(s) \cdot G(s) \)

Controller Sensitivity: \( K S_y = K(s) \cdot S_y(s) \)

\[ r(t) \rightarrow KS_y \]
\[ d_i(t) \rightarrow -T_u \]
\[ d_y(t) \rightarrow -KS_y \]
\[ n(t) \rightarrow -KS_y \]

\[ \sum \rightarrow u(t) \]

**Input performance**

\[ r(t) \rightarrow T_y \]
\[ d_i(t) \rightarrow S_y G \]
\[ d_y(t) \rightarrow S_y \]
\[ n(t) \rightarrow -T_y \]

\[ \sum \rightarrow y(t) \]

**Output performance**
Performance analysis and specification using the sensitivity functions: the MIMO case - Some classical analysis criteria (1)

- The transfer function $K S_y(s)$ should be upper bounded so that $u(t)$ does not reach the physical constraints, even for a large reference $r(t)$.

- The effect of the measurement noise $n(t)$ on the plant input $u(t)$ can be made « small » by making the sensitivity function $K S_u(s)$ small (in High Frequencies).

- The effect of the input disturbance $d_i(t)$ on the plant input $u(t) + d_i(t)$ (actuator) can be made « small » by making the sensitivity function $S_u(s)$ small (take care to not trying to minimize $T_u$ which is not possible).
Performance analysis and specification using the sensitivity functions: the MIMO case- Some classical analysis criteria (2)

- The plant output $y(t)$ can track the reference $r(t)$ by making the complementary sensitivity function $T_y(s)$ equal to 1. (servo pb)
- The effect of the output disturbance $d_y(t)$ (resp. input disturbance $d_i(t)$) on the plant output $y(t)$ can be made « small » by making the sensitivity function $S_y(s)$ (resp. $S_yG(s)$) « small »
- The effect of the measurement noise $n(t)$ on the plant output $y(t)$ can be made « small » by making the complementary sensitivity function $T_y(s)$ « small »

Output performance
Trade-offs

But

\[ S_\ast + T_\ast = I_\ast \]

Some trade-offs are to be looked for...

These trade-offs can be reached if one aims:

- to reject the disturbance effects in low frequencies
- to minimize the noise effects in high frequencies

It remains to require:

- \( S_y \) and \( S_yG \) to be small in low frequencies to reduce the load (output and input) disturbance effects on the controlled output
- \( T_y \) and \( KS_y \) to be small in high frequencies to reduce the effects of measurement noises on the controlled output and on the control input (actuator efforts)
4 Why $\mathcal{H}_\infty$ control is adapted to control engineering?

**Synthesis**

Provide a clear and detailed *frequency-domain* performance analysis using the sensitivity functions in order to explain the trade-off performance/robustness/actuator constraints.

**Qualitative analysis**

Use of $S_y, T_y, KS_y, S_yG$ to:

- predict the behavior of the **output** w.r.t different external inputs (reference, disturbance, noise)
- predict the behavior of the **control input** w.r.t different external inputs (reference, disturbance, noise)

**Quantitative analysis**

Use of $S_y, T_y, KS_y, S_yG$ to:

- Stability analysis and margins.
- Compute the steady-state errors in tracking, output and input disturbance attenuations.
- Give the maximum of the input/output gains to analyze the transient behaviors of the output and control input (incl. saturation).
- Give all the bandwidths of the sensitivity functions
- Evaluate the rise time in tracking
- Evaluate the robustness margins
A first insight into the performance specifications for the MIMO case

The direct extension of the performances objectives to MIMO systems could be formulated as follows:

1. **Disturbance attenuation/closed-loop performances:**
   
   $$\bar{\sigma}(S_y(j\omega)) < \frac{1}{|W_1(j\omega)|}$$

   with $|W_1(j\omega)| > 1$ for $\omega < \omega_b$

2. **Actuator constraints:**
   
   $$\bar{\sigma}(KS_y(j\omega)) < \frac{1}{|W_2(j\omega)|}$$

   with $|W_2(j\omega)| > 1$ for $\omega > \omega_h$

3. **Robustness to multiplicative uncertainties:**
   
   $$\bar{\sigma}(T_y(j\omega)) < \frac{1}{|W_3(j\omega)|}$$

   with $|W_3(j\omega)| > 1$ for $\omega > \omega_t$

However these objectives do not consider the specific MIMO structure of the system, i.e. the input-output relationship between actuators and sensors. It is then better to define the objectives accordingly with the system inputs and outputs.
Towards MIMO systems

Let us consider a system with 2 inputs and 1 output and define:

\[ G = \begin{pmatrix} G_1 & G_2 \end{pmatrix}, \quad K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \]

Therefore

\[ GK = G_1 K_1 + G_2 K_2, \quad KG = \begin{pmatrix} K_1G_1 & K_1G_2 \\ K_2G_1 & K_2G_2 \end{pmatrix} \]

and the sensitivity functions are:

\[ S_y = \frac{1}{1 + G_1 K_1 + G_2 K_2}, \quad KS_y = \begin{pmatrix} K_1 \\ \frac{1+G_1 K_1+G_2 K_2}{K_2} \\ \frac{1+G_1 K_1+G_2 K_2}{1+G_1 K_1+G_2 K_2} \end{pmatrix} \]

While a single template \( W_e \) is convenient for \( S_y \) it is straightforward that the following diagonal template should be used for \( KS_y \):

\[ W_u(s) = \begin{pmatrix} W_u^1(s) & 0 \\ 0 & W_u^2(s) \end{pmatrix} \]

where \( W_u^1 \) and \( W_u^2 \) are chosen in order to account for each actuator specificity (constraint).
The MIMO general case

Let us consider $G$ with $m$ inputs and $p$ outputs.

- In the MIMO case the simplest way is to defined the templates as diagonal transfer matrices, i.e. using $(M_{S_i}, \omega_{b_i}, \varepsilon_i)$
- In that case, a weighting function should be dedicated for each input, and for each output.
- These weighting functions may of course be different if the specifications on each actuator (e.g. saturation), and on each sensor (e.g. noise), are different.

In addition, during the performance analysis step, take care to plot, in addition to the MIMO sensitivity functions, the individual ones related to each input/output to check if the individual specification is met. Hence, in the simplest case:

1. If the specifications are identical then it is sufficient to plot:
   - $\bar{\sigma}(S_y(j\omega))$ and $\frac{1}{|W_e(j\omega)|}$, for all $\omega$
   - $\bar{\sigma}(KS_y(j\omega))$ and $\frac{1}{|W_u(j\omega)|}$, for all $\omega$

2. If the specifications are different, one should plot
   - $\bar{\sigma}(S_y(i,:))$ and $\frac{1}{|W_e^i|}$, for all $\omega$, $i = 1, \ldots, p$
   - $\bar{\sigma}(KS_y(k,:))$ and $\frac{1}{|W_u^k|}$, for all $\omega$, $k = 1, \ldots, m$

   i.e. $p$ plots for all output behaviors and $m$ plots for the input ones.

3. In a very general case, plot $\bar{\sigma}(S_y)$ with $\bar{\sigma}(1/W_e)$
More on weighting functions

When tighter (harder) objectives are to be met ..... the templates can be defined more accurately by transfer functions of order greater than 1, as

$$W_{e}(s) = \left(\frac{s/M_S + \omega b}{s + \omega_b \varepsilon}\right)^k,$$

if a roll-off of $-20 \times k$ dB per decade is required. Take care to the choice of the parameters $(M_S, \omega_b, \varepsilon)$ to avoid incoherent objectives!
Final objectives

In terms of control synthesis, all these specifications can be tackled in the following problem: find $K(s)$ s.t.

$$\begin{bmatrix}
W_e S_y \\
W_u K S_y \\
W_T T_y \\
W_{SG} S_y G
\end{bmatrix}_{\infty} \leq 1 \quad \Rightarrow \quad \|W_e S_y\|_{\infty} \leq 1 \quad \|W_u K S_y\|_{\infty} \leq 1 \quad \|W_G S_y G\|_{\infty} \leq 1 \quad \|W_T T_y\|_{\infty} \leq 1$$

Often, the simpler following one (referred to as the **mixed sensitivity problem**) is studied:

$$\text{Find } K \text{ s.t. } \begin{bmatrix}
W_e S_y \\
W_u K S_y
\end{bmatrix}_{\infty} \leq 1$$

since the latter allows to consider the closed-loop output performance as well as the actuator constraints.
4 Why $H_\infty$ control is adapted to control engineering?

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How to consider performance specification in $\mathcal{H}_\infty$ control?

Illustration on the $\mathcal{H}_\infty$ SISO problem: $\|T_{ew}(s)\|_\infty = \left\| \begin{array}{c} W_e S \\ W_u KS \end{array} \right\|_\infty \leq \gamma$

In that case the closed-loop system $T_{ew}(s)$ must have 1 input and 2 outputs. Since $S = \frac{r-y}{r}$ and $KS = \frac{u}{r}$, the control scheme needs only one external input $r$.

Control objectives:

$$y = Gu = GK(r - y) \Rightarrow \text{tracking error : } \varepsilon = Sr$$

$$u = K(r - y) = K(r - Gu) \Rightarrow \text{actuator force : } u = KSr$$

To cope with that control objectives the following control scheme is considered:

Objective w.r.t sensitivity functions: $\|W_e S\|_\infty \leq 1$, $\|W_u KS\|_\infty \leq 1$.

Idea: define 2 new virtual controlled outputs:

$$e_1 = W_e Sr$$
$$e_2 = W_u KSr$$
The mixed sensitivity $\mathcal{H}_\infty$ control design - Problem definition

The performance specifications on the tracking error & on the actuator, given as some weights on the controlled output, then leads to the new control scheme:

The associated general control configuration is:

\[ e = (e_1, e_2)^T \]

Controlled Outputs

Measured outputs
The corresponding $\mathcal{H}_\infty$ suboptimal control problem is therefore to find a controller $K(s)$ such that

$$\|T_{ew}(s)\|_\infty = \left\| \begin{bmatrix} W_eS \\ W_uKS \end{bmatrix} \right\|_\infty \leq \gamma$$

with

$$T_{ew}(s) = F_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

$$= \begin{bmatrix} W_e \\ 0 \end{bmatrix} + \begin{bmatrix} -W_eG \\ W_u \end{bmatrix} K(I + GK)^{-1}I$$

$$= \begin{bmatrix} W_eS \\ W_uKS \end{bmatrix}$$

in Matlab

```matlab
% Generalized plant P is found with function sysic
systemnames = 'G We Wu';
inputvar = ' [ r(1); u(1) ]';
outputvar = ' [ We; Wu; r-G ]';
input_to_G = ' [u]';
input_to_We = ' [r-G]';
input_to_Wu = ' [u]';
sysoutname = 'P';
cleanupsysic = 'yes';
sysic;
% Find H-infinity optimal controller
nmeas=1; nu=1;
[K,CL,GAM,INFO] = hinfsyn(P,nmeas,nu,'DISPLAY','ON');
gopt
```

O. Sename [GIPSA-lab]
What about disturbance attenuation?

To account for input disturbance rejection, the control scheme must include $d_i$:

This corresponds to the closed-loop system.

$$T_{ew} = \begin{bmatrix} W_e S_y & W_e S_y G \\ W_u K S_y & W_u T_u \end{bmatrix}$$

The new $\mathcal{H}_\infty$ control problem therefore includes the input disturbance rejection objective, thanks to $S_y G$ that should satisfy the same template as $S$, i.e an high-pass filter!

**Remarks:** Note that $W_u T_u$ is an additional constraint that may lead to an increase of the attenuation level $\gamma$ since it is not part of the objectives. Hopefully $T_u$ is low pass, and $W_u$ as well. The input weight has to be on $u$ not $u + d_i$ which would lead to an unsolvable problem.
Improve the disturbance attenuation

The previous problem, allows to ensure the input disturbance rejection, but does not provide any additional d-o-f to improve it (without impacting the tracking performance). In order to ‘decouple’ both performance objectives, the idea is to add a disturbance model that indeed changes the disturbance rejection properties.

Let then consider: \( d_i(t) = W_d . d \). In that case the closed-loop system is

\[
T_{ew} = \begin{bmatrix}
W_e S_y & W_e S_y G W_d \\
W_u K S_y & W_u T_u W_d
\end{bmatrix}
\]

and the template expected for \( S_y G \) is now \( \frac{1}{W_d . W_e} \).

First interest: improve the disturbance weight as \( W_d = 100 \) ... but this has a price (see Fig. below for an example)
More generally...

To include multiple objectives in a SINGLE $H_\infty$ control problem, there are 2 ways:

1. add some external inputs (reference, noise, disturbance, uncertainties ...)
2. add new controlled outputs

Of course both ways increase the dimension of the problem to be solved....thus the complexity as well. Moreover additional constraints appear that are not part of the objectives ....

General rule: first think simple !!
Some extensions: the 2-DOF case

In some cases it is interesting to decouple the transient response in tracking from the stabilization loop (as in RST controllers). This is the case of 2 dof control structure.

Pay attention when building $P$ since:
- External inputs: $r$, $d_i$, $d_y$ and $n$
- Control Input: $u$
- Controlled outputs $z_1$ and $z_2$
- Measurements: $r$ and $y + n$

The controller solution will be such as

$$u = \begin{bmatrix} K_r & K_y \end{bmatrix} \begin{bmatrix} r \\ -y - n \end{bmatrix}$$
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Introduction

Framework

Main extracts of this part: see Goodwin et al 2001. "Performance limitations in control are not only inherently interesting, but also have a major impact on real world problems."

Objective: take into account the limitations inherent to the system or due to actuators constraints, before designing the controller..... Understanding what is not possible is as important as understanding what is possible!

Example of structural constraints

\[ S + T = 1, \forall \omega \]

We then cannot have, for any frequency \( \omega_0 \), \( |S(j\omega_0)| < 1 \) and \( |T(j\omega_0)| < 1 \). This implies that, disturbance and noise rejection cannot be achieved in the same frequency range.

Bode’s Sensitivity Integral for open-loop stable systems

It is known that, for an open loop stable plant:

\[ \int_0^\infty \log|S(j\omega)| \, d\omega = 0 \]

Then the frequency range where \( |S(j\omega)| < 1 \) is balanced by the frequencies where \( |S(j\omega)| > 1 \).
Bode sensitivity

Nice interpretation of the balance between reduction and magnification of the sensitivity.

For open-loop unstable systems we have a stronger constraint:

$$\int_0^\infty \log |\det(S(j\omega))| \, d\omega = \pi \sum_{i=1}^{N_p} \text{Re}(p_i),$$

where $p_i$ design the $N_p$ RHP poles. Therefore, in the presence of RHP poles, the control effort necessary to stabilize the system is paid in terms of amplification of the sensitivity magnitude.
The interesting case of systems with RHP zeros

Theorem

Let $G(s)$ a MIMO plant with one RHP zero at $s = z$, and $W_p(s)$ be a scalar weight. Then, closed-loop stability is ensured only if:

$$\| W_p(s)S(s) \| \geq |W_p(s = z)|$$

To illustrate the use of that theorem, if $W_p$ is chosen as:

$$W_p(s) = \left( \frac{s/M_S + \omega b}{s + \omega b \varepsilon} \right),$$

and, if the controller meets the requirements, then

$$\| W_p(s)S(s) \|_{\infty} \leq 1$$

Therefore a necessary condition is:

$$\left| \frac{z/M_S + \omega b}{z + \omega b \varepsilon} \right| \leq 1$$

To conclude, if $z$ is real, and if the performance specifications are such that: $M_S = 2$ and $\varepsilon = 0$, then a necessary condition to meet the performance requirements is:

$$\omega_b \leq \frac{z}{2}$$
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A first case: the state feedback control problem

Let consider the system:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) \\
z(t) &= Cx(t) + D_{11} w(t) + D_{12} u(t)
\end{align*}
\]  

(20)

The objective is to find a state feedback control law \( u = -Kx \) s.t:

\[
\|Tzw(s)\|_\infty \leq \gamma
\]

The method consists in applying the Bounded Real Lemma to the closed-loop system, and then try to obtain some convex solutions (LMI formulation).

This is achieved if and only is there exists a positive definite symmetric matrix \( P \) (i.e \( P = P^T > 0 \) s.t

\[
\begin{bmatrix}
(A - B_2 K)^T P + P (A - B_2 K) & P B_1 & C^T \\
* & -\gamma I & D^T \\
* & * & -\gamma I
\end{bmatrix}
\] < 0, \( P > 0 \).

(21)
Solution of the state feedback control problem

Use of change of variables

First, left and right multiplication by $\text{diag}(P^{-1}, I_n, I_n)$, and use $Q = P^{-1}$ and $Y = -KP^{-1}$. It leads to

$$\begin{bmatrix}
AQ + B_2 Y + QA^T + YT B_2^T & B_1 & QC^T - YT D_{12}^T \\
* & -\gamma I & D_{11}^T \\
* & * & -\gamma I
\end{bmatrix} < 0, \quad Q > 0. \quad (22)$$

The state feedback controller is then:

$$K = -YQ^{-1}$$
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The Dynamic Output feedback case

It will be shown how to formulate such a control problem using "classical" control tools. The procedure will be 2-steps:

**Get P:** Build the General Control Configuration scheme s.t. the closed-loop system matrix does correspond to the tackled $H_\infty$ problem (for instance the mixed sensitivity problem). Use of Matlab, sysic tool. A state space representation of $P$, the generalized plant, is needed.

**Compute K:** Use an optimisation algorithm that finds the controller $K$ solution of the considered problem.

The calculation of the controller, solution of the $H_\infty$ control problem, can then be done using the Riccati approach or the LMI approach of the $H_\infty$ control problem [Zhou et al.(1996)Zhou, Doyle, and Glover] [Skogestad and Postlethwaite(1996)].

Notations:

\[
P \begin{cases}
\dot{x} &= Ax + B_1 w + B_2 u \\
z &= C_1 x + D_{11} w + D_{12} u \\
y &= C_2 x + D_{21} w + D_{22} u
\end{cases} \Rightarrow P = \begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix}
\]

$x \in \mathbb{R}^n$: \{ plant state variables $\cup$ state variables of weights}  
with $w \in \mathbb{R}^{n_w}$: external inputs  
$z \in \mathbb{R}^{n_z}$: controlled outputs  
$u \in \mathbb{R}^{n_u}$ control inputs  
y $\in \mathbb{R}^{n_y}$ measured outputs (inputs of the controller)
Problem formulation

Let $K(s)$ be a dynamic output feedback LTI controller defined as

$$K(s) : \begin{cases} \dot{x}_K(t) &= A_K x_K(t) + B_K y(t), \\ u(t) &= C_K x_K(t) + D_K y(t). \end{cases}$$

where $x_K \in \mathbb{R}^n$, and $A_K, B_K, C_K$ and $D_K$ are matrices of appropriate dimensions.

**Remark.** The controller will be considered here of the same order (same number of state variables) $n$ than the generalized plant, which here, in the $H_\infty$ framework, the order of the optimal controller.

With $P(s)$ and $K(s)$, the closed-loop system $N(s)$ is:

$$N(s) : \begin{cases} \dot{x}_{cl}(t) &= A_{CL} x_{cl}(t) + B_{CL} w(t), \\ z(t) &= C_{CL} x_{cl}(t) + D_{CL} w(t), \end{cases} \tag{23}$$

where $x_{cl}^T(t) = [x^T(t) \ x_K^T(t)]$ and

$$\begin{align*} A_{CL} &= \begin{pmatrix} A + B_2 D_K & C_2 \\ B_K C_2 & A_K \end{pmatrix}, \\
B_{CL} &= \begin{pmatrix} B_1 + B_2 D_K & D_{21} \\ B_K D_{21} & A_K \end{pmatrix}, \\
C_{CL} &= \begin{pmatrix} C_1 + D_{12} D_K & C_2, \\ D_{12} C_K \end{pmatrix}, \\
D_{CL} &= B_1 + B_2 D_K D_{21}. \end{align*}$$

The aim is of course to find matrices $A_K, B_K, C_K$ and $D_K$ s.t. the $H_\infty$ norm of the closed-loop system (23) is as small as possible, i.e. $\gamma_{opt} = \min \gamma$ s.t. $\|N(s)\|_\infty < \gamma$. 
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Assumptions for the Riccati method

A1: \((A, B_2)\) stabilizable and \((C_2, A)\) detectable: necessary for the existence of stabilizing controllers

A2: \(\text{rank}(D_{12}) = n_u\) and \(\text{rank}(D_{21}) = n_y\): Sufficient to ensure the controllers are proper, hence realizable

A3: \(\forall \omega \in \mathbb{R}, \text{rank}\left(\begin{array}{cc} A - j\omega I_n & B_2 \\ C_1 & D_{12} \end{array}\right) = n + n_u\)

A4: \(\forall \omega \in \mathbb{R}, \text{rank}\left(\begin{array}{cc} A - j\omega I_n & B_1 \\ C_2 & D_{21} \end{array}\right) = n + n_y\) Both ensure that the optimal controller does not try to cancel poles or zeros on the imaginary axis which would result in CL instability

A5:
\[
D_{11} = 0, \quad D_{22} = 0, \quad D_{12}^T\begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I_{n_u} \end{bmatrix},
\]
\[
\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I_{n_y} \end{bmatrix}
\]
not necessary but simplify the solution (does correspond to the given theorem next but can be easily relaxed)
The problem solvability

The first step is to check whether a solution does exist of not, to the optimal control problem.

Theorem (1)

Under the assumptions A1 to A5, there exists a dynamic output feedback controller

\[ u(t) = K(.) \ y(t) \]

such that the closed-loop system is internally stable and the \( H_\infty \) norm of the closed-loop system from the exogenous inputs \( w(t) \) to the controlled outputs \( z(t) \) is less than \( \gamma \), if and only if

i. the Hamiltonian \( H = \begin{pmatrix} A & \gamma^{-2} B_1 B^T_1 - B_2 B^T_2 \\ -C_1^T & C_1 \end{pmatrix} \) has no eigenvalues on the imaginary axis.

ii. there exists \( X_\infty \geq 0 \) t.q. \( A^T \ X_\infty + X_\infty \ A + X_\infty \ (\gamma^{-2} B_1 B^T_1 - B_2 B^T_2) \ X_\infty + C_1^T C_1 = 0 \),

iii. the Hamiltonian \( J = \begin{pmatrix} A^T & \gamma^{-2} C_1^T C_1 - C_2^T C_2 \\ -B_1 B^T_1 \end{pmatrix} \) has no eigenvalues on the imaginary axis.

iv. there exists \( Y_\infty \geq 0 \) t.q. \( A \ Y_\infty + Y_\infty \ A^T + Y_\infty \ (\gamma^{-2} C_1^T C_1 - C_2^T C_2) \ Y_\infty + B_1 B^T_1 = 0 \),

v. the spectral radius \( \rho(X_\infty \ Y_\infty) \leq \gamma^2 \).
Controller reconstruction

Theorem (2)

If the necessary and sufficient conditions of the Theorem 1 are satisfied, then the so-called central controller is given by the state space representation

\[ K_{\text{sub}}(s) = \begin{bmatrix} \hat{A}_\infty & -Z_\infty L_\infty \\ F_\infty & 0 \end{bmatrix} \]

with

\[ \hat{A}_\infty = A + \gamma^{-2} B_1 B_1^T X_\infty + B_2 F_\infty + Z_\infty L_\infty C_2 \]
\[ F_\infty = -B_2^T X_\infty, \quad L_\infty = -Y_\infty C_2^T \]
\[ Z_\infty = (I - \gamma^{-2} Y_\infty X_\infty)^{-1} \]

The Controller structure is indeed an observer-based state feedback control law, with

\[ u_2(t) = -B_2^T X_\infty \hat{x}(t), \]

where \( \hat{x}(t) \) is the observer state vector

\[ \dot{\hat{x}}(t) = A \hat{x}(t) + B_1 \hat{w}(t) + B_2 u(t) + Z_\infty L_\infty \left( C_2 \hat{x}(t) - y(t) \right). \quad (24) \]

and \( \hat{w}(t) \) is defined as

\[ \hat{w}(t) = \gamma^{-2} B_1^T X_\infty \hat{x}(t). \]

Remark. \( \hat{w}(t) \) is an estimation of the worst case disturbance. \( Z_\infty L_\infty \) is the filter gain for the OE problem of estimating \( \hat{x}(t) \) in the presence of the worst case disturbance.
5 How to solve an $H_\infty$ control problem?

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   - The $H_\infty$ norm definition
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6. The LMI approach for $H_\infty$ control design

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The LMI approach for $\mathcal{H}_\infty$ control design- Solvability

In this case only $A_1$ is necessary. The solution is based on the use of the Bounded Real Lemma, and some relaxations that lead to an LMI problem to be solved [Scherer(1990)].

When we refer to the $\mathcal{H}_\infty$ control problem, we mean: Find a controller $K$ for system $P$ such that, given $\gamma_\infty$,

$$\|\mathcal{F}_l(P, K)\|_\infty < \gamma_\infty$$

The minimum of this norm is denoted as $\gamma^*_\infty$ and is called the optimal $\mathcal{H}_\infty$ gain. Hence, it comes,

$$\gamma^*_\infty = \min_{(A_K, B_K, C_K, D_K)} \|T_{zw}(s)\|_\infty$$

As presented in the previous sections, this condition is fulfilled thanks to the BRL. As a matter of fact, the system is internally stable and meets the quadratic $\mathcal{H}_\infty$ performances iff. $\exists \mathcal{P} = \mathcal{P}^T > 0$ such that,

$$
\begin{bmatrix}
A_{CL}^T \mathcal{P} + \mathcal{P} A_{CL} & \mathcal{P} B_{CL} & C_{CL}^T \\
B_{CL}^T \mathcal{P} & -\gamma_2^2 I & D_{CL}^T \\
C_{CL} & D_{CL} & -I
\end{bmatrix} < 0
$$

(27)

where $A_{CL}, B_{CL}, C_{CL}, D_{CL}$ are given in (23). Since this inequality is not an LMI and not tractable for SDP solver, relaxations have to be performed (indeed it is a BMI), as proposed in [Scherer et al.(1997b)Scherer, Gahinet, and Chilali].
The LMI approach for $\mathcal{H}_\infty$ control design - Problem solution

Theorem (LTI/$\mathcal{H}_\infty$ solution [Scherer et al.(1997a)]

A dynamical output feedback controller of the form $K(s) = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$ that solves the $\mathcal{H}_\infty$ control problem, is obtained by solving the following LMIs in ($X$, $Y$, $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$), while minimizing $\gamma_\infty$,

$$
\begin{bmatrix}
M_{11} & (*)^T & (*)^T & (*)^T \\
M_{21} & M_{22} & (*)^T & (*)^T \\
M_{31} & M_{32} & M_{33} & (*)^T \\
M_{41} & M_{42} & M_{43} & M_{44}
\end{bmatrix} \prec \begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \succ 0
$$

(28)

where,

$$
M_{11} = AX + XA^T + B_2 \tilde{C} + \tilde{C}^T B_2^T \\
M_{22} = YA + A^T Y + \tilde{B} C_2 + C_2^T \tilde{B}^T \\
M_{32} = B_1^T Y + D_{21}^T \tilde{B}^T \\
M_{44} = C_1 X + D_{12} \tilde{C} \\
M_{43} = D_{11} + D_{12} \tilde{D} D_{21}
$$

$$
M_{21} = \tilde{A} + A^T + C_2^T \tilde{D}^T B_2^T \\
M_{31} = B_1^T + D_{21}^T \tilde{D} B_2^T \\
M_{33} = -\gamma_\infty I_{n_u} \\
M_{42} = C_1 + D_{12} \tilde{D} C_2 \\
M_{44} = -\gamma_\infty I_{n_y}
$$

(29)
Controller reconstruction

Once $A$, $B$, $C$, $D$, $X$ and $Y$ have been obtained, the reconstruction procedure consists in finding non singular matrices $M$ and $N$ s.t. $M N^T = I - X Y$ and the controller $K$ is obtained as follows

$$
\begin{align*}
D_K &= \tilde{D} \\
C_K &= (\tilde{C} - D_c C_2 X) M^{-T} \\
B_K &= N^{-1} (\tilde{B} - Y B_2 D_c) \\
A_K &= N^{-1} (\tilde{A} - Y A X - Y B_2 D_c C_2 X - N B_c C_2 X - Y B_2 C_c M^T) M^{-T}
\end{align*}
$$

(30)

where $M$ and $N$ are defined such that $M N^T = I_n - X Y$ (that can be solved through a singular value decomposition plus a Cholesky factorization).

**Remark.** Note that other relaxation methods can be used to solve this problem, as suggested by [Gahinet(1994)].
6 Robust analysis

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Introduction

- A control system is robust if it is insensitive to differences between the actual system and the model of the system which was used to design the controller.
- How to take into account the difference between the actual system and the model?
- A solution: using a model set BUT: very large problem and not exact yet.

**A method:** these differences are referred as model uncertainty.

The approach:

1. Determine the uncertainty set: mathematical representation
2. Check Robust Stability
3. Check Robust Performance

Lots of forms can be derived according to both our knowledge of the physical mechanism that cause the uncertainties and our ability to represent these mechanisms in a way that facilitates convenient manipulation.

Several origins:

- Approximate knowledge and variations of some parameters
- Measurement imperfections (due to sensor)
- At high frequencies, even the structure and the model order is unknown (100)
- Choice of simpler models for control synthesis
- Controller implementation

Two classes: parametric uncertainties / neglected or unmodelled dynamics.
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Example 1: uncertainties

Let consider the example from (Sokestag & Postlewaite, 1996).

\[ \tilde{G}(s) = \frac{k}{1 + \tau s} e^{-sh}, \quad 2 \leq k, \ h, \ \tau \leq 3 \]

Let us choose the nominal parameters as, \( k = h = \tau = 2.5 \) and \( G \) the according nominal model. We can define the ‘relative’ uncertainty, which is actually referred as a MULTIPLICATIVE UNCERTAINTY, as

\[ \tilde{G}(s) = G(s)(I + W_m(s)\Delta(s)) \]

with \( W_m(s) = \frac{3.5s + 0.25}{s + 1} \)

and \( \|\Delta\|_\infty \leq 1 \)
Example 2: unmodelled dynamics

Let us consider the system:

\[ \tilde{G}(s) = G_0(s) \frac{1}{1 + \tau s}, \quad \tau \leq \tau_{\text{max}} \]

This can be modelled as:

\[ \tilde{G}(s) = G_0(s)(I + W_m(s)\Delta(s)) \]

with \( W_m(s) = \frac{\tau_{\text{max}}j\omega}{1+\tau_{\text{max}}j\omega} \) and \( \|\Delta\|_\infty \leq 1 \)

This can be represented as
Example 3: parametric uncertainties

Consider the first order system:

\[ G(s) = \frac{1}{s + a}, \quad a_0 - b < a < a_0 + b \]

Define now:

\[ a = a_0 + \delta.b \quad \text{with} \quad |\delta| < 1 \]

Then it leads:

\[ \frac{1}{s + a} = \frac{1}{s + a_0 + \delta.b} = \frac{1}{s + a_0} \left(1 + \frac{\delta.b}{s + a_0}\right)^{-1} \]

This can then be represented as a Multiplicative Inverse Uncertainty:
Example 3 (cont.) same example with state space formulation

Let us first the transfer function \( G(s) = \frac{1}{s+a} \) as

\[
G : \begin{cases}
\dot{x} = (-a_0 - \delta.b)x + w \\
z = x
\end{cases}
\]  \hspace{1cm} (31)

In order to use an LFT, let us define the uncertain input:

\[
u_\Delta = \delta x,
\]

Then the previous system can be rewritten in the following LFR:

where \( \Delta \) and \( y_\Delta \) are given as:

\[
\Delta = [\delta], \quad y_\Delta = (x)
\]

and \( N \) given by the state space representation:

\[
N : \begin{cases}
\dot{x} = -a_0 x - b u_\Delta + w \\
y_\Delta = x \\
z = x
\end{cases}
\]  \hspace{1cm} (32)
Example 4: parametric uncertainties in state space equations

Let us consider the following uncertain system:

\[
G : \begin{cases}
\dot{x}_1 &= (-2 + \delta_1)x_1 + (-3 + \delta_2)x_2 \\
\dot{x}_2 &= (-1 + \delta_3)x_2 + u \\
y &= x_1
\end{cases}
\]

(33)

In order to use an LFT, let us define the uncertain inputs:

\[
u_{\Delta_1} = \delta_1 x_1, \quad u_{\Delta_2} = \delta_2 x_2, \quad u_{\Delta_3} = \delta_3 x_2
\]

Then the previous system can be rewritten in the following LFR:

where \(\Delta\) and \(y_{\Delta}\) are given as:

\[
\Delta = \begin{bmatrix}
\delta_1 & 0 & 0 \\
0 & \delta_2 & 0 \\
0 & 0 & \delta_3
\end{bmatrix}, \quad y_{\Delta} = \begin{pmatrix} x_1 \\ x_2 \\ x_2 \end{pmatrix}
\]

and \(N\) given by the state space representation:

\[
N : \begin{cases}
\dot{x}_1 &= -2x_1 - 3x_2 + u_{\Delta_1} + u_{\Delta_2} \\
\dot{x}_2 &= -x_2 + u + u_{\Delta_3} \\
y &= x_1
\end{cases}
\]
Towards LFR (LFT)

The previous computations are in fact the first step towards an unified representation of the uncertainties: the Linear Fractional Representation (LFR).
Indeed the previous schemes can be rewritten in the following general representation as:

$$F_u(N, \Delta) = N_{22} + N_{21} \Delta (I - N_{11} \Delta)^{-1} N_{12}$$

This LFT exists and is well-posed if $$(I - N_{11} \Delta)^{-1}$$ is invertible.
LFT definition

In this representation $N$ is known and $\Delta(s)$ collects all the uncertainties taken into account for the stability analysis of the uncertain closed-loop system. $\Delta(s)$ shall have the following structure:

$$\Delta(s) = \text{diag} \{ \Delta_1(s), \cdots, \Delta_q(s), \delta_1 I_r, \cdots, \delta_r I_r, \epsilon_1 I_c, \cdots, \epsilon_c I_c \}$$

with $\Delta_i(s) \in \mathcal{RH}_\infty^{k_i \times k_i}$, $\delta_i \in \mathbb{R}$ and $\epsilon_i \in \mathbb{C}$.

**Remark:** $\Delta(s)$ includes

- $q$ full block transfer matrices,
- $r$ real diagonal blocks referred to as 'repeated scalars' (indeed each block includes a real parameter $\delta_i$ repeated $r_i$ times),
- $c$ complex scalars $\epsilon_i$ repeated $c_i$ times.

**Constraints:** The uncertainties must be normalized, i.e such that:

$$\|\Delta\|_\infty \leq 1, \quad |\delta_i| \leq 1, \quad |\epsilon_i| \leq 1$$
Uncertainty types

We have seen in the previous examples the two important classes of uncertainties, namely:

- **UNSTRUCTURED UNCERTAINTIES**: we ignore the structure of $\Delta$, considered as a full complex perturbation matrix, such that $\|\Delta\|_\infty \leq 1$.
  
  We then look at the maximal admissible norm for $\Delta$, to get Robust Stability and Performance. This will give a global sufficient condition on the robustness of the control scheme. This may lead to conservative results since all uncertainties are collected into a single matrix ignoring the specific role of each uncertain parameter/block.

- **STRUCTURED UNCERTAINTIES**: we take into account the structure of $\Delta$, (always such that $\|\Delta\|_\infty \leq 1$).
  
  The robust analysis will then be carried out for each uncertain parameter/block. This needs to introduce a new tool: the **Structured Singular Value**. We then can obtain more fine results but using more complex tools.

The analysis is provided in what follows for both cases. In **Matlab** this analysis is provided in the tools `robuststab` and `robustperf`. 
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Robustness analysis: problem formulation

Since the analysis will be carried out for a closed-loop system, \( N \) should be defined as the connection of the plant and the controller. Therefore, in the framework of the \( H_\infty \) control, the following extended General Control Configuration is considered:

\[
\begin{align*}
\Delta & \quad u_\Delta \\
\quad & \quad P \\
\quad & \quad K \\
\quad & \quad y \\
\quad & \quad e \\
\quad & \quad w \\
\text{External inputs} & \quad \text{Controlled outputs} \\
\text{Control input} & \quad \text{Measured outputs}
\end{align*}
\]

Figure: \( P - K - \Delta \) structure

and \( N \) is such that

\[
N = F_l(P, K)
\]
Robust analysis: problem definition

In the global $P - K - \Delta$ General Control Configuration, the transfer matrix from $w$ to $z$ (i.e the closed-loop uncertain system) is given by:

$$z = F_u(N, \Delta)w,$$

with $F_u(N, \Delta) = N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}$.

and the objectives are then formulated as follows:

**Nominal stability (NS):** $N$ is internally stable

**Nominal Performance (NP):** $\|N_{22}\|_\infty < 1$ and NS

**Robust stability (RS):** $F_u(N, \Delta)$ is stable $\forall \Delta$, $\|\Delta\|_\infty < 1$ and NS

**Robust performance (RP):** $\|F_u(N, \Delta)\|_\infty < 1$ $\forall \Delta$, $\|\Delta\|_\infty < 1$ and NS
Towards Robust stability analysis

**Robust Stability**= with a given controller $K$, we determine whether the system remains stable for all plants in the uncertainty set.

According to the definition of the previous upper LFT, when $N$ is stable, the instability may only come from $(I - N_{11} \Delta)$. Then it is equivalent to study the $M - \Delta$ structure, given as:

$$\begin{align*}
\Delta \\
\Delta \\
\Delta
\end{align*}$$

This leads to the definition of the Small Gain Theorem

**Theorem (Small Gain Theorem)**

Suppose $M \in RH_{\infty}$. Then the closed-loop system in Fig. 7 is well-posed and internally stable for all $\Delta \in RH_{\infty}$ such that:

$$\|\Delta\|_{\infty} \leq \delta (\text{resp. } < \delta) \quad \text{if and only if} \quad \|M(s)\|_{\infty} < 1/\delta (\text{resp. } \|M(s)\|_{\infty} \leq 1)$$
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O. Sename [GIPSA-lab]
Definition of the uncertainty types

**Additive**

\[ W_A(s) \rightarrow \Delta_A(s) \rightarrow G(s) \rightarrow y \]

**Additive inverse**

\[ u_\Delta \rightarrow \Delta_{iA}(s) \rightarrow W_{iA}(s) \rightarrow y \]

**Output Multiplicative**

\[ u \rightarrow G(s) \rightarrow W_0(s) \rightarrow \Delta_0(s) \rightarrow u_\Delta \rightarrow y \]

**Input Multiplicative**

\[ u \rightarrow G(s) \rightarrow W_1(s) \rightarrow \Delta_I(s) \rightarrow u_\Delta \rightarrow y \]

**Output Inverse Multiplicative**

\[ u \rightarrow G(s) \rightarrow \Delta_{iO}(s) \rightarrow W_{iO}(s) \rightarrow y \]

**Input Inverse Multiplicative**

\[ u \rightarrow G(s) \rightarrow \Delta_{iI}(s) \rightarrow W_{iI}(s) \rightarrow y \]

*Figure: 6 uncertainty representations*
Robust stability analysis: additive case

Objective: applying the Small Gain Theorem to these unstructured uncertainty representations.

Let us consider the following simple control scheme as:

\[ r(t) + \varepsilon(t) \rightarrow u(t) \rightarrow K(s) \rightarrow G(s) \rightarrow y(t) \]

Additive case:
\[ \tilde{G}(s) = G(s) + W_A(s) \Delta_A(s). \]
Computing the \( N - \Delta \) form gives
\[ N(s) = \begin{pmatrix} -W_A K S_y & W_A K S_y \\ S_y & T_y \end{pmatrix} \]

The objective is to obtain:

Output Multiplicative uncertainties:
\[ \tilde{G}(s) = (I + W_O(s) \Delta_O(s)) G(s). \]
Then it leads
\[ N(s) = \begin{pmatrix} -W_O T_y & W_O T_y \\ S_y & T_y \end{pmatrix} \]
General results

Theorem (Small Gain Theorem)

Consider the different uncertainty types, and assume that NS is achieved, i.e. $M \in RH_\infty$ for each type. Then the closed-loop system is robustly stable, i.e. internally stable for all $\Delta_k \in RH_\infty$ (for $k = A, 0, I, iO, iI$) such that:

- **Additive**: $\| W_A KS_y \|_\infty \leq 1$
- **Additive Inverse**: $\| W_{iA} S_y \|_\infty \leq 1$
- **Output Multiplicative**: $\| W_O T_y \|_\infty \leq 1$
- **Input Multiplicative**: $\| W_I T_u \|_\infty \leq 1$
- **Output Inverse Multiplicative**: $\| W_{iO} S_y \|_\infty \leq 1$
- **Input Inverse Multiplicative**: $\| W_{iI} S_u \|_\infty \leq 1$

This gives some robustness templates for the sensitivity functions. However this may be conservative.
Illustration on the SISO case

Here Robust Stability is analyzed through the Nyquist plot. For illustration, let us consider the case of Multiplicative uncertainties (Input and Output case are identical for SISO systems), i.e

$$\tilde{G} = G(I + W_m \Delta_m)$$

Then the loop transfer function is given as:

$$\tilde{L} = \tilde{G}K = GK(I + W_m \Delta_m) = L + W_m L \Delta_m;$$

According to the Nyquist theorem, RS is achieved the the closed-loop system is stable for any $\tilde{L}$ should not encircle, i.e $\tilde{L}$ should not encircle -1 for all uncertainties. According to the figure, a sufficient condition is then:

$$|W_m L| < |1 + L|, \forall \omega$$

$$\Leftrightarrow \left| \frac{W_m L}{1+L} \right| < 1, \forall \omega$$

$$\Leftrightarrow |W_m T| < 1 \forall \omega$$
A first insight in Robust Performance

Objective: applying the Small Gain Theorem to these unstructured uncertainty representations.

Let us consider the following simple control scheme as:

\[
\begin{align*}
\dot{e}(t) &= W_e(s) e(t) \\
K(s) u(t) &= \tilde{G}(s) e(t) + \Delta_0(s) G(s)
\end{align*}
\]

Case of **Output Multiplicative** uncertainties:
\[
\tilde{G}(s) = (I + W_O(s) \Delta_O(s)) G(s).
\]

Computing the \( N - \Delta \) form gives

\[
N(s) = \begin{bmatrix}
N_{11}(s) & N_{12}(s) \\
N_{21}(s) & N_{22}(s)
\end{bmatrix}
= \begin{bmatrix}
-W_OT_y & W_OT_y \\
-W_eS_y & W_eS_y
\end{bmatrix}
\]

The objectives are then formulated as follows:

**NS:** \( N \) is internally stable

**NP:** \( \| W_eS_y \|_\infty < 1 \) and NS

**RS:** \( \| W_OT_y \|_\infty < 1 \) and NS

**RP:** \( \| F_u(N, \Delta) \|_\infty < 1 \) \( \forall \Delta \), \( \| \Delta \|_\infty < 1 \),

Sufficient condition: NS and
\[
\bar{\sigma}(W_OT_y) + \bar{\sigma}(W_eS_y) < 1, \ \forall \omega
\]
Illustration on the SISO case

Here Robust Performance is analyzed through the Nyquist plot. For illustration, let us consider the case of Multiplicative uncertainties (Input and Output case are identical for SISO systems), i.e

\[ \tilde{G} = G(I + W_m \Delta_m) \]

Then the loop transfer function is given as:

\[ \tilde{L} = \tilde{G}K = GK(I + W_m \Delta_m) = L + W_m L \Delta_m; \]

First NP is achieved when:

\[ |W_e S| < 1 \quad \forall \omega, \quad \Leftrightarrow \quad |W_e| < |1 + L|, \quad \forall \omega. \]

Therefore RP is achieved if

\[ |W_e \tilde{S}| < 1, \quad \forall \tilde{S}, \forall \omega \]

\[ \Leftrightarrow \quad |W_e| < |1 + \tilde{L}|, \quad \forall \tilde{L}, \forall \omega \]

Since \(|1 + \tilde{L}| \geq |1 + L| - |W_m L \Delta_m|\), a sufficient condition is actually:

\[ |W_e| + |W_m L| < |1 + L|, \quad \forall \omega \]

\[ \Leftrightarrow \quad |W_e S| + |W_m T| < 1, \quad \forall \omega \]
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The structured case

\[ \Delta = \{ \text{diag}\{\Delta_1, \cdots, \Delta_q, \delta_1 I_{r_1}, \cdots, \delta_r I_{r_r}, \epsilon_1 I_{c_1}, \cdots, \epsilon_c I_{c_c} \} \in \mathbb{C}^{k \times k} \} \] (35)

with \( \Delta_i \in \mathbb{C}^{k_i \times k_i}, \delta_i \in \mathbb{R}, \epsilon_i \in \mathbb{C}, \)

where \( \Delta_i(s), i = 1, \ldots, q, \) represent full block complex uncertainties, \( \delta_i(s), i = 1, \ldots, r, \) real parametric uncertainties, and \( \epsilon_i(s), i = 1, \ldots, c, \) complex parametric uncertainties.

Taking into account the uncertainties leads to the following General Control Configuration,

Figure: General control configuration with uncertainties

where \( \Delta \in \Delta. \)
The structured singular value

To handle parametric uncertainties, we need to introduce $\mu$, the structured singular value, defined as:

**Definition ($\mu$)**

For $M \in \mathbb{C}^{n \times n}$, the structure singular value is defined as:

$$
\mu_\Delta(M) := \frac{1}{\min \{\sigma(\Delta) : \Delta \in \Delta, \det(I - \Delta M) \neq 0\}}
$$

In other words, it allows to find the smallest structured $\Delta$ which makes $\det(I - M\Delta) = 0$.

**Theorem (The structured Small Gain Theorem)**

Let $M(s)$ be a MIMO LTI stable system and $\Delta(s)$ a LTI uncertain stable matrix, (i.e. $\in \mathcal{RH}_\infty$). The system in Fig. 7 is stable for all $\Delta(s)$ in (35) if and only if:

$$
\forall \omega \in \mathbb{R} \quad \mu_\Delta(M(j\omega)) \leq 1, \text{ with } M(s) := N_{Zv}(s)
$$

More generally both following statements are equivalent

- For $\bar{\mu} \in \mathbb{R}$, $N(s)$ and $\Delta(s)$ belong to $\mathcal{RH}_\infty$, and

$$
\forall \omega \in \mathbb{R}, \quad \mu_\Delta(M(j\omega)) \leq \bar{\mu}
$$

- the system represented in figure 7 is stable for any uncertainty $\Delta(s)$ of the form (35) such that:

$$
||\Delta(s)||_{\mathcal{H}_\infty} < 1/\bar{\mu}
$$
Fictive uncertainties: full complex matrix representing the $H_\infty$ norm specifications

Real uncertainties: block diagonal matrix

Disturbances & references $w$ 
Control input $u$ 
Measured output $y$ 
Controlled outputs $e$
Introduction of a fictive block

Usually only real parametric uncertainties (given in $\Delta_r$) are considered for RS analysis. RP analysis also needs a fictive full block complex uncertainty, as below,

\[
\begin{bmatrix}
\Delta(s) \\
\Delta f \\
\Delta_r \\
\end{bmatrix}
\]

\[
N(s) = \begin{bmatrix}
N_{11}(s) & N_{12}(s) \\
N_{21}(s) & N_{22}(s) \\
\end{bmatrix}, \quad \text{and the closed-loop transfer matrix is:}
\]

\[
T_{ew}(s) = N_{22}(s) + N_{21}(s)\Delta(s)(I - N_{11}(s))^{-1}N_{12}(s)
\]
For RS, we shall determine how large $\Delta$ (in the sense of $H_\infty$) can be without destabilizing the feedback system. From (36), the feedback system becomes unstable if $\det(I - N_{11}(s)) = 0$ for some $s \in \mathbb{C}$, $\Re(s) \geq 0$. The result is then the following.

**Theorem ([?])**

Assume that the nominal system $N_{ew}$ and the perturbations $\Delta$ are stable. Then the feedback system is stable for all allowed perturbations $\Delta$ such that $||\Delta(s)||_\infty < 1/\beta$ if and only if $\forall \omega \in \mathbb{R}$, $\mu_{\Delta}(N_{11}(j\omega)) \leq \beta$.

Assuming nominal stability, RS and RP analysis for structured uncertainties are therefore such that:

\[
\text{NP} \iff \sigma(N_{22}) = \mu_{\Delta_f}(N_{22}) \leq 1, \ \forall \omega
\]

\[
\text{RS} \iff \mu_{\Delta_r}(N_{11}) < 1, \ \forall \omega
\]

\[
\text{RP} \iff \mu_{\Delta}(N) < 1, \ \forall \omega, \ \Delta = \begin{bmatrix} \Delta_f & 0 \\ 0 & \Delta_r \end{bmatrix}
\]

Finally, let us remark that the structured singular value cannot be explicitly determined, so that the method consists in calculating an upper bound and a lower bound, as closed as possible to $\mu$. 

O. Sename [GIPSA-lab]
Summary

The steps to be followed in the RS/RP analysis for structured uncertainties are then:

- Definition of the real uncertainties $\Delta_r$ and of the weighting functions
- Evaluation of $\mu(N_{22}) \Delta_f$, $\mu(N_{11}) \Delta_r$ and $\mu(N) \Delta$
- Computation of the admissible intervals for each parameter

Remark: The Robust Performance analysis is quite conservative and requires a tight definition of the weighting functions that do represent the performance objectives to be satisfied by the uncertain closed-loop system. Therefore it is necessary to distinguish the weighting functions used for the nominal design from the ones used for RP analysis.
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Brief overview

In order to design a robust control, i.e. a controller for which the synthesis actually accounts for uncertainties, some of the methods are:

- **Unstructured uncertainties**: Consider an uncertainty weight (unstructured form), and include the Small Gain Condition through a new controlled output. For example, robustness face to Output Multiplicative Uncertainties can be considered into the design procedure adding the controlled output $e_y = W_O y$, which, when tracking performance is expected, leads to the condition $\| W_O T_y \|_\infty \leq 1$.

- **Structured uncertainties**: the design of a robust controller in the presence of such uncertainties is the $\mu$-synthesis. It is handled through an interactive procedure, referred to as the $DK$ iteration. This procedure is much more involved than a "simple" $H_\infty$ control design and often leads to an increase of the order of the controller (which is already the sum of the order of the plant and of the weighting functions).

- Use other mathematical representation of parametric uncertainties, [Scherer and Wieland(2004)], as for instance the **polytopic model**. In that case the set of uncertain parameters is assumed to be a polytope (i.e. a convex) set. The stability issue in that framework is referred to as the 'Quadratic stability' i.e find a single Lyapunov function for the uncertainty set. While in the general case this is an unbounded problem, in the polytopic case (or in the affine case), the stability is to be analyzed only at the vertices of the polytope, which is a finite dimensional problem.

This approach can then be applied to find a single controller, valid over the potyopic set. Note that this approach gives rise to the LPV design for polytopic systems, as described next.
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\( \mathcal{H}_2 \) design

The \( \mathcal{H}_\infty \) norm considered above gives the system gain when input and output are measured using the \( L_2 \) norm. Rather than bounding the output energy, it may be desirable to keep the peak amplitude of the controlled output below a certain level, e.g. to avoid actuator saturations. Now, when we refer to the \( \mathcal{H}_2 \) control problem, we mean: Find a controller \( C \) for system \( M \) such that, given \( \gamma_\infty \),

\[
|| F_l(M, C) ||_2 < \gamma_2 \tag{37}
\]

The \( \mathcal{H}_2 \) problem can be expressed as follow

\[
\begin{bmatrix}
A^T K + KA & KB \\
B^T K & -I
\end{bmatrix} < 0 \quad \begin{bmatrix}
K & C^T \\
C & Z
\end{bmatrix} > 0, \text{Trace}(Z) < \gamma_2
\]

where \( A, B, C, D \) are given in (??).
**H\(\infty\)/H\(2\) problem formulation**

Useful to deal with different objectives functions of the external signal types (noise, disturbance..).

The resulting LMI based problem formulation consists in solving the following problem subject to \(K = K^T \succ 0\) (note that to obtain LMIs, the same change of variable as introduced in the \(H\infty\) and \(H\)2 problems can be applied).

Objectives:
\[
T_\infty = \left\| \begin{bmatrix} z_w \nabla w \end{bmatrix} \right\|_{\infty} < \gamma_\infty \\
T_2 = \left\| \begin{bmatrix} z_2 \nabla w_2 \end{bmatrix} \right\|_{2} < \gamma_2
\]

Even after the change of basis, it is impossible (non convex problem) to minimize simultaneously the \(H\infty\) and \(H\)2 criteria. As a consequence, the problem is usually reformulated as one of the problems below:

- A linear combination of \(\gamma_\infty\) and \(\gamma_2\), e.g.: \(\gamma_{mix} = \alpha \gamma_\infty + (1 - \alpha) \gamma_2\), where \(\alpha \in [0 1]\)

- Minimize \(\gamma_\infty\) (resp. \(\gamma_2\)) while fixing \(\gamma_2\) (resp. \(\gamma_\infty\)).
**$\mathcal{H}_\infty$ observer definition**

Let consider the system:

$$
\begin{align*}
\dot{x}(t) &= Ax(t) + Ew(t) + Bu(t) \\
y(t) &= Cx(t) + Fw(t)
\end{align*}
$$

(39)

The objective is

$$
\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t))
$$

(40)

where $\hat{x}(t) \in \mathbb{R}^n$ is the estimated state of $x(t)$ and $L$ is the $n \times p$ constant observer gain matrix to be designed.

The estimated error, $e(t) := x(t) - \hat{x}(t)$, satisfies:

$$
\dot{e}(t) = (A - LC)e(t) + (E - LF)w(t)
$$

(41)

**Problem definition**

System (75) is said to be a $H_\infty$ observer for the above system if:

$$
\lim_{t \to \infty} e(t) \to 0 \quad \text{for} \quad w(t) \equiv 0
$$

(42)

$$
\|T_{ew}(s)\|_\infty \leq \gamma \quad \text{under} \quad \hat{e}(t = 0) = 0
$$

(43)
\( \mathcal{H}_\infty \) observer design

The method consists (as for state feedback design) to apply the BRL to the error equation and use some change of variables to get some LMIs. This is achieved if and only if there exists a positive definite symmetric matrix \( P \) (i.e \( P = P^T > 0 \) s.t

\[
\begin{bmatrix}
(A - LC)^T P + P (A - LC) & P (E - LF) & I_n \\
* & -\gamma I & 0 \\
* & * & -\gamma I
\end{bmatrix} < 0, \quad P > 0.
\] (44)

Use of change of variables

Let define \( Y = -KP \). It leads to

\[
\begin{bmatrix}
AP + YC + PA^T + C^TY^T & PE + YF & I_n \\
* & -\gamma I & 0 \\
* & * & -\gamma I
\end{bmatrix} < 0, \quad Q > 0.
\] (45)

The observer gain is then:

\[
L = -P^{-1}Y
\]
Other interests of the $\mathcal{H}_\infty$ approach

Control

Using LMI s the previous methods can be designed to take into account

- **Pole placement constraints**: useful to avoid fast dynamics and high frequencies in the controller (to facilitate digital implementation).

- **Input and state constraints**: some results allow to included together with $\mathcal{H}_\infty$ performance, the saturation constraints on the input (to provide an anti-windup scheme) (and state constraints)

- **Passivity performance**: used to enforce dissipative properties of the closed loop (this property is widely used in e.g. electrical systems, robotic applications). This property ensures that the introduced energy is dissipated into the system. This approach is linked with the passivity theory.

Observer design, Fault Diagnosis and Fault Tolerant Control

- Design of $\mathcal{H}_\infty$ observer and robust observers.

- Design $\mathcal{H}_\infty$ observers for Fault Detection and Isolation (FDI) (sometimes using a bank of observers) and for Fault Estimation as well.

- Reconfiguration of (state or dynamic output) feedback control: The controller changes according to detected faults

Last be not least: all what has been seen in the course does exist for discrete-time systems.
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LPV systems

Definition of a Linear Parameter Varying system

\[
\Sigma(\rho) : \begin{bmatrix}
\dot{x} \\
z \\
y
\end{bmatrix} = \begin{bmatrix}
A(\rho) & B_1(\rho) & B_2(\rho) \\
C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \\
C_2(\rho) & D_{21}(\rho) & D_{22}(\rho)
\end{bmatrix} \begin{bmatrix}
x \\
w \\
u
\end{bmatrix}
\]

\(x(t) \in \mathbb{R}^n\), ...., \(\rho = (\rho_1(t), \rho_2(t), \ldots, \rho_N(t)) \in \Omega\), is a vector of time-varying parameters (\(\Omega\) convex set), assumed to be known \(\forall t\)
LPV systems

Definition of an Linear Parameter Varying system

\[ \Sigma(\rho) : \begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A(\rho) & B_1(\rho) & B_2(\rho) \\ C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \\ C_2(\rho) & D_{21}(\rho) & D_{22}(\rho) \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \]

\( x(t) \in \mathbb{R}^n, \ldots, \rho = (\rho_1(t), \rho_2(t), \ldots, \rho_N(t)) \in \Omega, \) is a vector of time-varying parameters (\( \Omega \) convex set), assumed to be known \( \forall t \)

(Damper mass-spring system:
\[ \ddot{p} + c \dot{p} + k(t) p = u, \quad y = x \]

First-order state-space representation:
\[ \frac{d}{dt} \begin{bmatrix} p \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k(t) & -c \end{bmatrix} \begin{bmatrix} p \\ \dot{p} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \]
\[ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p \\ \dot{p} \end{bmatrix} \]
LPV systems

Definition of a Linear Parameter Varying system

\[
\Sigma(\rho) : \begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A(\rho) & B_1(\rho) & B_2(\rho) \\ C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \\ C_2(\rho) & D_{21}(\rho) & D_{22}(\rho) \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix}
\]

\( x(t) \in \mathbb{R}^n, \ldots, \rho = (\rho_1(t), \rho_2(t), \ldots, \rho_N(t)) \in \Omega \), is a vector of time-varying parameters (\( \Omega \) convex set), assumed to be known \( \forall t \).

The frozen Bode plots for \( c = 1 \) and \( k \in [1, 3] \) (Scherer, ACC Tutorial 2012)

Dampened mass-spring system:

\[ \ddot{p} + c \dot{p} + k(t) p = u, \quad y = x \]

First-order state-space representation:

\[ \frac{d}{dt} \begin{pmatrix} p \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k(t) & -c \end{pmatrix} \begin{pmatrix} p \\ \dot{p} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \quad y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} p \\ \dot{p} \end{pmatrix} \]
LPV systems (2)

Let the LPV system be:

\[
\Sigma(\rho) : \begin{bmatrix}
\dot{x} \\
z \\
y
\end{bmatrix} = \begin{bmatrix}
A(\rho) & B_1(\rho) & B_2(\rho) \\
C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \\
C_2(\rho) & D_{21}(\rho) & D_{22}(\rho)
\end{bmatrix}
\begin{bmatrix}
x \\
w \\
u
\end{bmatrix}
\]

\(x(t) \in \mathbb{R}^n, \ldots, \rho = (\rho_1(t), \rho_2(t), \ldots, \rho_N(t)) \in U_{\rho},\) is a vector of time-varying parameters \((U_{\rho} \text{ convex set})\)

- \(\rho(.)\) varies in the set of continuously differentiable parameter curves \(\rho : [0, \infty) \rightarrow \mathbb{R}^N.\)
  It is assumed to be **known or measurable**.
- The parameters \(\rho\) are always assumed to be **bounded**:
  \[
  \rho \in U_{\rho} \subset \mathbb{R}^N \text{ and } U_{\rho} \text{ compact}
  \]
  defined by the minimal \(\rho_i\), and maximal \(\overline{\rho_i}\) values of \(\rho_i(t)\)
  \[
  \rho_i(t) \in [\rho_i, \overline{\rho_i}], \ \forall i
  \]
- The system matrices \(A(.) \ldots\) are continuous on \(U_{\rho}\)
LPV systems (3): about the parameters

- Parameters are **exogenous** if they are external variables. The system is in that case *non stationary*. See the previous damped mass-spring system.

- Parameters are **endogenous** if they are function of the state variables, $\rho = \rho(x(t), t)$, and, in that case, the LPV system is referred to as a **quasi-LPV system**. This case is encountered when approximating Nonlinear systems. For instance:

  \[
  \dot{x}(t) = x^2(t) = \rho(t)x(t)
  \]

  with $\rho(t) = x(t)$.

- It is sometimes required that the derivative of the parameters are bounded, i.e:

  \[
  \dot{\rho} \in U_\dot{\rho} \subset \mathbb{R}^N \text{ and } U_\dot{\rho} \text{ compact}
  \]

  defined by the minimal $\nu_i$, and maximal $\overline{\nu_i}$ values of $\dot{\rho}_i(t)$

  \[
  \dot{\rho}_i(t) \in [\nu_i, \overline{\nu_i}], \ \forall i
  \]

  This corresponds to the case of *slow varying parameters*

- Other representations can be considered if $\rho$ is piecewise-constant, or varies in a finite set of elements ($\rho(t) \in \{0, 1\}$ for switching systems)

Next, several classes of LPV models are presented, and some ways to go from one class to another are given.
Some comments

- LPV systems can model uncertain systems ($\rho$ fixed but unknown) or parameter-varying models ($\rho(t)$)

LPV=linear or nonlinear?

- What is often referred to as gain-scheduling control, corresponds to Jacobian linearization of the nonlinear plant about a family of equilibrium points Shamma (90), Rugh & Shamma (2000)
  - In terms of control design this means, linearization around operating conditions, design (at each operating points) of a LTI controller, and interpolation of the LTI controllers in between operating conditions (often used in Aerospace and Automotive industries).
  - **Pros**: Simplicity of design for a non-linear system
  - **Cons**: No *a priori* guarantee of stability nor robustness

- **But**: this differs from quasi-LPV representations where nonlinearities are hidden in some parameter descriptions (as seen later in the course)

LPV=LTV

- Theoretical analysis of LPV system properties (stability, controllability, observability), often falls into the framework of LTV systems or of nonlinear ones (for quasi-LPV representations), see (Blanchini).
Some references

Those not to be ignored

- Modelling, identification: (Bruzelius, Bamieh, Lovera, Toth) + 2011 TCST Special Issue on "Applied LPV modelling and identification"
- Control (Shamma, Apkarian & Gahinet, Adams, Packard, Beker, Seiler, Grigoriadis ...)
- Stability, stabilization (Scherer, Wu, Blanchini ...)
- Geometric analysis (Bokor & Balas)
- Survey paper: Hoffmann & Werner, 2015
- Fault tolerant control: special issues by
  - Casavola, Rodrigues & Theilliol, 2015: in International Journal of Robust and Nonlinear Control

Some recent books

- R. Toth, Modeling and identification of linear parameter-varying systems, Springer 2010
Class 1: Affine parameter dependence

In this case, the system matrices are such that:

\[ A(\rho) = A_0 + A_1 \rho_1 + ... + A_N \rho_N \]

This is the case of the damped mass-spring system:

\[ \ddot{p} + c\dot{p} + k(t)p = u \]

considering the state space representation

\[
\begin{align*}
\dot{x}(t) &= A(k(t))x(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

with \( x(t) \in \mathbb{R}^2 \) and

\[
A(k(t)) = \begin{pmatrix} 0 & 1 \\ -k(t) & -c \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad D = 0
\]

Denoting the varying parameter \( \rho(t) = k(t) \), we get:

\[
A_0 = \begin{pmatrix} 0 & 1 \\ 0 & -c \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}
\]
Class 2: Polynomial parameter dependence

In this case, the system matrices are such that:

\[ A(\rho) = A_0 + A_1 \rho + A_2 \rho^2 + \ldots + A_S \rho^S \]

Example 1

Let consider for instance the following example in (Briat, 2015):

\[ \dot{x}(t) = x^3(t) \]

which can be written as

\[ \dot{x}(t) = -\rho(t)^2 x(t), \text{ with } \rho(t) = x(t) \]

Example 2

Another example is the sampling-dependent discrete-time system representation (Robert et al, 2010):

\[ G_d : \begin{cases} 
  x_{k+1} = A_d(h) x_k + B_d(h) u_k \\
  y_k = C(h) x_k + D u_k 
\end{cases} \quad (49) \]

where \( A_d(h) = e^{Ah} \) is approximated using Taylor expansions, such as:

\[ A_d(h) \approx I + \sum_{i=1}^{N} \frac{A^i}{i!} h^i \]
Class 3: Polytopic models

A polytopic system is represented as

$$\Sigma(\rho) = \sum_{k=1}^{Z} \alpha_k(\rho) \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}, \quad \text{with} \quad \sum_{k=1}^{2N} \alpha_k(\rho) = 1, \alpha_k(\rho) > 0$$

where $$\begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}$$ are LTI systems.

This representation is often used to rewrite an affine LPV system. Indeed, assuming that the parameters are bounded ($$\rho_i \in [\rho_i, \overline{\rho_i}]$$), the vector of parameters evolves inside a polytope represented by $$Z = 2^N$$ vertices $$\omega_i$$, as

$$\rho \in \text{Co}\{\omega_1, \ldots, \omega_Z\}$$  \hspace{1cm} (50)

It is then written as the convex combination:

$$\rho = \sum_{i=1}^{Z} \alpha_i \omega_i, \quad \alpha_i \geq 0, \quad \sum_{i=1}^{Z} \alpha_i = 1$$  \hspace{1cm} (51)

where the vertices are defined by a vector $$\omega_i = [\nu_{i1}, \ldots, \nu_{iN}]$$ where $$\nu_{ij}$$ equals $$\rho_j$$ or $$\overline{\rho_j}$$.

The LTI system $$\begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}$$ here corresponds to the LPV system frozen at the vertex $$k$$. 

O. Sename [GIPSA-lab]
Nonlinear vs Linear Differential Inclusion

See (Boyd et al, 1994).
Let consider the nonlinear system

\[ \Sigma_{NL} : \begin{cases} \dot{x} = f(x(t), w(t)) \\ z = g(x(t), w(t)) \end{cases} \tag{52} \]

Suppose that, for each \( x, w \) and \( t \), there is a matrix \( G(x, w, t) \in \Omega \) s.t.:

\[
\begin{bmatrix} f(x, w) \\ g(x, w) \end{bmatrix} = G(x, w, t) \begin{bmatrix} x \\ w \end{bmatrix} \tag{53}
\]

where \( \Omega \in \mathbb{R}^{(n_x+n_z) \times (n_x+n_u)} \).
As said in (Boyd et al, 1994):
"Then of course every trajectory of the nonlinear system (52) is also a trajectory of the LDI defined by (53). If we can prove that every trajectory of the LDI defined by (53) has some property (e.g., converges to zero), then a fortiori we have proved that every trajectory of the nonlinear system (52) has this property."
Example of a one-tank model - MATLAB session 1

Usually the hydraulic equation is non linear and of the form

\[ S \frac{dH}{dt} = Q_e - Q_s \]

where \( H \) is the tank height, \( S \) the tank surface, \( Q_e \) the input flow, and \( Q_s \) the output flow defined by \( Q_s = k_t \sqrt{H} \).

Definition the state space model

The system is represented by an Ordinary Differential Equation whose solution depends on \( H(t_0) \) and \( Q_e \). Clearly \( H \) is the system state, \( Q_e \) is the input, and the system can be represented as:

\[
\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0
\]  

(54)

with \( x = H \), \( f = -\frac{k_t}{S} \sqrt{x} + \frac{1}{S} u \)

A LPV model

Following the LDI method, we define:

\[ \rho(x) = \frac{\sqrt{x}}{x} \]

which leads to the LPV model

\[
\dot{x}(t) = -\frac{k_t}{S} \cdot \rho(x) \cdot x + \frac{1}{S} u, \quad x(0) = x_0
\]  

(55)
Example of a one-tank model - MATLAB session 1 (cont.)

Consider the variations $q_e, q_s, h$ around the steady state as:

$Q_e = Q_0 + q_e; \quad Q_s = Q_0 + q_s; \quad H = H_0 + h.$

This leads to the equation:

$$S \frac{dh}{dt} = q_e - k_t \left( \sqrt{H_0 + h} - \sqrt{H_0} \right)$$

Denoting the state variable $x = h$, the control input $u = q_e$, the output $y = h$, the tangent linearization gives

$$\dot{x} = Ax + Bu \quad (56)$$
$$y = Cx \quad (57)$$

with $A = -\frac{k_t}{2S\sqrt{H_0}}$, $B = \frac{1}{S}$ and $C = 1$.

Now implement 3 types of models and compare with the nonlinear one:

- The linear model around a fixed operating point
- A LPV model obtained defining an adequate LDI
- A LPV model defined in the LFR framework
LPV systems properties

Let consider the LPV system

\[
\Sigma_\rho \left\{ \begin{array}{l}
\dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))u(t), \quad x(0) = x_0 \\
y(t) = C(\rho(t))x(t) + D(\rho(t))u(t)
\end{array} \right.
\] (58)

What kind of properties we should pay attention to?

When \( \rho \) is fixed (constant) the previous system is LTI and

- controllability, observability, stability, are uniquely defined
- controllability \( \iff \) reachability, observability \( \iff \) reconstructibility
- these properties are equivalent by a state change of basis.

But when \( \rho(t) \) is time varying .....  

- these facts may not be true (asymptotic and exponential stability may differ)
- need to study properties of Linear \textit{Time-Varying} systems.
- A generalization of the \( \exp(At) \) is needed, defining the state transition matrix \( \Phi(t, t_0, \rho(t)) \)
- For a change of basis \( T(t) \) with \( x(t) = T(t)x_{\text{new}}(t) \) then, \( \dot{x}(t) = \dot{T}(t)x_{\text{new}}(t) + T(t)\dot{x}_{\text{new}}(t) \)
Problem statement and facts

Recall

For LTI systems all notions of stability are equivalent: global/local, asymptotic/exponential, time-domain (Lyapunov)/frequency-domain (Bode, poles...).

Why stability analysis for LPV systems is not an easy task?

Let consider $\dot{x} = A(\rho(t))x$. Stability analysis is more involved (as for LTV systems) since:

- there is a set of solutions for a given $x_0$ (family of systems from $\rho$ variations)
- the system may be stable for frozen parameter values and unstable for varying parameters (as for switching systems)
- asymptotic and exponential stability are no more equivalent and cannot be characterized by the eigenvalues of $A(\rho(t))$.
- In term of design, we will often rely on the notion of quadratic stability (using quadratic Lyapunov function $V(x) = x^TPX$) which is stronger but easier to check for stability and simpler to use for control and observer design, see (Wu, PhD 95)

Robust or LPV? (Blanchini,00 & 07)

- Robust analysis and control: dedicated to LTI systems subject to time-varying uncertainties
- LPV (or gain-scheduling) analysis and control: dedicated to LTV systems or to linearizations of non linear systems along the trajectory of $\rho$
Quadratic stability for time-varying parameters

Let us consider the LPV system

\[ \dot{x} = A(\rho(t))x \]

where \( \rho(t) \) is an time-varying parameter vector that belongs to an uncertainty set \( \Omega \).

Use of a single Lyapunov function

If there exists \( P = P^T > 0 \) such that:

\[ A(\rho(t))^T P + PA(\rho(t)) < 0, \forall \rho(t) \in \Omega \]

then the system is stable for arbitrarily fast time-varying uncertainties

Remarks

- Quadractic stability imples exponential stability (Wu, 95)
- It is an infinite dimension problem (can be relaxed for polytopic uncertainties)
- It could be conservative since stability is checked for any variation of the parameters!

Pay attention in what follows: LPV system means TIME-VARYING parameters so a polytopic LPV system is not an uncertain polytopic system (in the latter case the coefficient \( \alpha_i \) of the polytopic description are constant even if unknown)
\( \mathcal{L}_2 \) stability of LPV systems (Wu, 95)

**Definition**

Given a parametrically dependent stable LPV system \( \Sigma_\rho = (A(\rho), B(\rho), C(\rho), D(\rho)) \) for zero initial conditions \( x_0 \). The induced \( \mathcal{L}_2 \) norm is defined as:

\[
||\Sigma_\rho||_{i,2} = \sup_{\rho(t) \in \Omega} \sup_{w(t) \neq 0 \in \mathcal{L}_2} \frac{||y||_2}{||u||_2}
\]

which is often referred to as (by abuse of language) the \( H_\infty \) gain \( ||\Sigma_\rho||_\infty \) of the LPV system.

**Theorem**

A sufficient condition for the \( \mathcal{L}_2 \) stability of system \( \Sigma_\rho \) is the generalized BRL, using parameter dependent Lyapunov functions, i.e assuming \( |\dot{\rho}_i| < \nu_i, \forall i \), if there exists \( P(\rho) > 0, \forall \rho \) s.t

\[
\begin{bmatrix}
A(\rho)^T P(\rho) + P(\rho) A(\rho) + \sum_{i=1}^{N} \nu_i \frac{\partial P(\rho)}{\partial \rho_i} & P(\rho) B(\rho) & C(\rho)^T \\
B(\rho)^T P(\rho) & -\gamma I & D(\rho)^T \\
C(\rho) & D(\rho) & -\gamma I
\end{bmatrix} < 0, \forall i.
\] (59)

then \( ||\Sigma_\rho||_{i,2} \leq \gamma \)
Towards LPV control

The "gain scheduling" approach

Some references

- Modelling, identification: (Bruzelius, Bamieh, Lovera, Toth)
- Control (Shamma, Apkarian & Gahinet, Adams, Packard, Beker ...)
- Stability, stabilization (Scherer, Wu, Blanchini ...)
- Geometric analysis (Bokor & Balas)
State feedback control design: pole placement (1)

Let consider the system $\Sigma(\rho)$:

\[
\begin{align*}
\dot{x}(t) &= A(\rho)x(t) + B(\rho)u(t) \\
y(t) &= C(\rho)x(t) + D(\rho)u(t)
\end{align*}
\]  

(60)

The objective is to find a state feedback control law $u = -F(\rho)x + G(\rho)r$, where $r$ is the reference signal s.t:

- the closed-loop system is stable
- the output $y$ tracks the reference $r$ (unit closed-loop gain $y/r$)

The closed-loop system is

\[
\begin{align*}
\dot{x}(t) &= (A(\rho) - B(\rho)F(\rho))x(t) + B(\rho)G(\rho)r \\
y(t) &= (C(\rho)x(t) - D(\rho)F(\rho))x(t) + D(\rho)G(\rho)r
\end{align*}
\]  

(61)

Then we must consider the following issues:

- What controllability property shall we consider?
- What parameter dependency should we define for $(F(\rho), G(\rho))$?
- How to choose the poles (dynamics) of the closed-loop system?
State feedback control design: pole placement (2)

Here we can tackle several problems:

Robust SF control:
- Design a nominal state feedback control \((F, G)\) (for \(\Sigma(\rho_{\text{nominal}})\))
- Check quadratic stability and performances of the closed-loop system, with \((A(\rho) - B(\rho)F)\) ...

LPV SF control with fixed performances:
- Choose some desired poles \((p_1, p_2, \ldots, p_n)\) for the CL system
- Design \(F(\rho)\) such that \(\text{eig}(A(\rho) - B(\rho)F(\rho)) = (p_1, p_2, \ldots, p_n)\). Could be analytic design or 'frozen-type'.
- Check stability (quadratic?) of the CL system when \(\rho\) is time-varying

LPV SF control with varying (adaptive) performances:
- Choose some desired poles \((p_1(\rho), p_2(\rho), \ldots, p_n(\rho))\) for the CL system
- Design \(F(\rho)\) such that \(\text{eig}(A(\rho) - B(\rho)F(\rho)) = (p_1(\rho), p_2(\rho), \ldots, p_n(\rho))\). Could be analytic design or 'frozen-type'.
- Check stability (quadratic?) of the CL system when \(\rho\) is time-varying
- This latter case allows to schedule the performances according to the parameter changes, so to handle the trade-off, function of the parameter variations, between closed-loop system transient dynamics and control cost limitations
The $H_\infty$ state feedback control problem

Let consider the system:

$$
\dot{x}(t) = A(\rho)x(t) + B_1(\rho)w(t) + B_2(\rho)u(t) \\
y(t) = C(\rho)x(t) + D_{11}(\rho)w(t) + D_{12}(\rho)u(t)
$$

(62)

The objective is to find a state feedback control law $u = -K(\rho)x$ s.t:

$$
\|T_{yw}(s)\|_\infty \leq \gamma
$$

The method consists in applying the Bounded Real Lemma to the closed-loop system, and then try to obtain some convex solutions (LMI formulation).

Following the framework of quadratic stabilty, this is achieved if and only is there exists a positive definite symmetric matrix $P$ (i.e $P = P^T > 0$ s.t

$$
\begin{bmatrix}
(A(\rho) - B_2(\rho)K(\rho))^T P + P (A(\rho) - B_2(\rho)K(\rho)) & P B_1(\rho) & (C(\rho) - D_{12}(\rho)K(\rho))^T \\
* & -\gamma I & D_{11}(\rho)^T \\
* & * & -\gamma I
\end{bmatrix} < 0.
$$

(63)
Solution of the state feedback control problem

Use of change of variables

First, left and right multiplication by \( \text{diag}(P^{-1}, I_n, I_n) \), and use \( Q = P^{-1} \) and \( Y(\rho) = -K(\rho)P^{-1} \). It leads to \( Q > 0 \) and

\[
\begin{bmatrix}
A(\rho)Q + B_2(\rho)Y(\rho) + Q A^T(\rho) + Y(\rho)^T B_2^T(\rho) & B_1(\rho) & QC^T(\rho) - Y^T(\rho) D_{12}^T(\rho) \\
\star & -\gamma I & D_{11}^T(\rho) \\
\star & \star & -\gamma I
\end{bmatrix} < 0. \tag{64}
\]

The state feedback controller is then:

\[
K(\rho) = -Y(\rho) Q^{-1}
\]

How to solve (64)?

- It is indeed not an LMI in \( \rho \) due to \( B_2(\rho)Y(\rho) \) and to \( D_{12}(\rho)Y(\rho) \), and still infinite dimensional
- For polytopic systems, a solution exists if either we choose \( Y \) time-invariant or if \( B_2 \) and \( D_{12} \) are parameter independent
- In the general case this requires to impose some parameter dependency on \( Y(\rho) \) (affine, polynomial ...) and to solve the problems trying to linearize it (or using gridding techniques (Balasz, Packard, Seiler)).
The $H_\infty/LPV$ control problem

**Definition**

Find a LPV controller $C(\rho)$ s.t the closed-loop system is stable and for $\gamma_\infty > 0$, $\sup \|z\|^2 < \gamma_\infty$,

- Unbounded set of LMIs (Linear Matrix Inequalities) to be solved ($\rho \in \Omega$)
- **Some approaches**: polytopic, LFT, gridding. See Arzelier [HDR, 2005], Bruzelius [Thesis, 2004], Apkarian et al. [TAC, 1995]...

**A solution: The "polytopic" approach [C. Scherer et al. 1997]**

- Problem solved off line for each vertex of a polytope (convex optimisation) (using here a single Lyapunov function i.e. quadratic stabilization).
- On-line the controller is computed as the convex combination of local linear controllers

$$C(\rho) = \sum_{k=1}^{2N} \alpha_k(\rho) \begin{bmatrix} A_c(\omega_k) & B_c(\omega_k) \\ C_c(\omega_k) & D_c(\omega_k) \end{bmatrix}, \sum_{k=1}^{2N} \alpha_k(\rho) = 1, \alpha_k(\rho) > 0$$

- Easy implementation !!
The $H_\infty/\text{LPV}$ control problem

Definition

Find a LPV controller $C(\rho)$ s.t the closed-loop system is stable and for $\gamma_\infty > 0$, $sup \|z\|_2 \|w\|_2 < \gamma_\infty$,

- Unbounded set of LMIs (Linear Matrix Inequalities) to be solved ($\rho \in \Omega$)

A solution: The "polytopic" approach [C. Scherer et al. 1997]

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$$C(\rho) = \sum_{k=1}^{2N} \alpha_k(\rho) \begin{bmatrix} A_c(\omega_k) & B_c(\omega_k) \\ C_c(\omega_k) & D_c(\omega_k) \end{bmatrix}, \sum_{k=1}^{2N} \alpha_k(\rho) = 1, \alpha_k(\rho) > 0$$

- Easy implementation !!
LPV control design

Dynamical LPV generalized plant:

\[
\Sigma(\rho) : \begin{bmatrix}
\dot{x} \\
z \\
y
\end{bmatrix}
= \begin{bmatrix}
A(\rho) & B_1(\rho) & B_2(\rho) \\
C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \\
C_2(\rho) & D_{21}(\rho) & D_{22}(\rho)
\end{bmatrix}
\begin{bmatrix}
x \\
w \\
u
\end{bmatrix}
\tag{65}
\]

LPV controller structure:

\[
S(\rho) : \begin{bmatrix}
\dot{x}_c \\
u
\end{bmatrix}
= \begin{bmatrix}
A_c(\rho) & B_c(\rho) \\
C_c(\rho) & D_c(\rho)
\end{bmatrix}
\begin{bmatrix}
x_c \\
y
\end{bmatrix}
\tag{66}
\]

LPV closed-loop system:

\[
\mathcal{CL}(\rho) : \begin{bmatrix}
\dot{\xi} \\
z
\end{bmatrix}
= \begin{bmatrix}
A(\rho) & B(\rho) \\
C(\rho) & D(\rho)
\end{bmatrix}
\begin{bmatrix}
\xi \\
w
\end{bmatrix}
\tag{67}
\]
LPV control design

Dynamical LPV generalized plant:

\[ \Sigma(\rho) : \begin{bmatrix} \dot{x} \\
\begin{array}{c} z \\ y \end{array} \end{bmatrix} = \begin{bmatrix} A(\rho) & B_1(\rho) & B_2(\rho) \\
C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \\
C_2(\rho) & D_{21}(\rho) & D_{22}(\rho) \end{bmatrix} \begin{bmatrix} x \\
w \end{bmatrix} \]  

(65)

LPV controller structure:

\[ S(\rho) : \begin{bmatrix} \dot{x}_c \\
u \end{bmatrix} = \begin{bmatrix} A_c(\rho) & B_c(\rho) \\
C_c(\rho) & D_c(\rho) \end{bmatrix} \begin{bmatrix} x_c \\
y \end{bmatrix} \]  

(66)

LPV closed-loop system:

\[ \mathcal{CL}(\rho) : \begin{bmatrix} \dot{\xi} \\
\begin{array}{c} z \\ w \end{array} \end{bmatrix} = \begin{bmatrix} A(\rho) & B(\rho) \\
C(\rho) & D(\rho) \end{bmatrix} \begin{bmatrix} \xi \\
w \end{bmatrix} \]  

(67)
LPV control design

Dynamical LPV generalized plant:

\[
\Sigma(\rho) : \begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A(\rho) & B_1(\rho) & B_2(\rho) \\ C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \\ C_2(\rho) & D_{21}(\rho) & D_{22}(\rho) \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \tag{65}
\]

LPV controller structure:

\[
S(\rho) : \begin{bmatrix} \dot{x}_c \\ u \end{bmatrix} = \begin{bmatrix} A_c(\rho) & B_c(\rho) \\ C_c(\rho) & D_c(\rho) \end{bmatrix} \begin{bmatrix} x_c \\ y \end{bmatrix} \tag{66}
\]

LPV closed-loop system:

\[
CL(\rho) : \begin{bmatrix} \dot{\xi} \\ z \end{bmatrix} = \begin{bmatrix} A(\rho) & B(\rho) \\ C(\rho) & D(\rho) \end{bmatrix} \begin{bmatrix} \xi \\ w \end{bmatrix} \tag{67}
\]
LPV control design

Dynamical LPV generalized plant:

\[
\Sigma(\rho) : \begin{bmatrix}
\dot{x} \\
\frac{z}{y}
\end{bmatrix} = \begin{bmatrix}
A(\rho) & B_1(\rho) & B_2(\rho) \\
C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \\
C_2(\rho) & D_{21}(\rho) & D_{22}(\rho)
\end{bmatrix} \begin{bmatrix}
x \\
w
\end{bmatrix}
\]  

(65)

LPV controller structure:

\[
S(\rho) : \begin{bmatrix}
\dot{x}_c \\
u
\end{bmatrix} = \begin{bmatrix}
A_c(\rho) & B_c(\rho) \\
C_c(\rho) & D_c(\rho)
\end{bmatrix} \begin{bmatrix}
x_c \\
y
\end{bmatrix}
\]  

(66)

LPV closed-loop system:

\[
\mathcal{CL}(\rho) : \begin{bmatrix}
\dot{\xi} \\
\frac{z}{y}
\end{bmatrix} = \begin{bmatrix}
A(\rho) & B(\rho) \\
C(\rho) & D(\rho)
\end{bmatrix} \begin{bmatrix}
\xi \\
w
\end{bmatrix}
\]  

(67)
LPV control design

$\mathcal{H}_\infty$ criteria Apkarian et al. [TAC, 1995]

Stabilize system $CL(\rho)$ (find $K > 0$) while minimizing $\gamma_\infty$.

$$\begin{bmatrix}
A(\rho)^T K + K A(\rho) & KB_\infty(\rho) & C_\infty(\rho)^T \\
B_\infty(\rho)^T K & -\gamma_\infty^2 I & D_\infty(\rho)^T \\
C_\infty(\rho) & D_\infty(\rho) & -I
\end{bmatrix} < 0$$

Infinite set of LMIs to solve $(\rho \in \Omega)$ ($\Omega$ is convex)

LFT, Gridding, Polytopic
LPV control design

$H_\infty$ criteria Apkarian et al. [TAC, 1995]

Stabilize system $CL(\rho)$ (find $K > 0$) while minimizing $\gamma_\infty$.

$$\begin{bmatrix}
A(\rho)^T K + K A(\rho) & KB_\infty(\rho) & C_\infty(\rho)^T \\
B_\infty(\rho)^T K & -\gamma_\infty^2 I & D_\infty(\rho)^T \\
C_\infty(\rho) & D_\infty(\rho) & -I
\end{bmatrix} < 0$$

Infinite set of LMIs to solve $(\rho \in \Omega)$ ($\Omega$ is convex)


LFT, Gridding, Polytopic
LPV control design

Polytopic approach

Solve the LMIs at each vertex of the polytope formed by the extremum values of each varying parameter, with a common $K$ Lyapunov function.

\[
C(\rho) = \sum_{k=1}^{2^N} \alpha_k(\rho) \begin{bmatrix} A_c(\omega_k) & B_c(\omega_k) \\ C_c(\omega_k) & D_c(\omega_k) \end{bmatrix}
\]

where,

\[
\alpha_k(\rho) = \frac{\prod_{j=1}^{N} |\rho_j - C_c(\omega_k)_j|}{\prod_{j=1}^{N} (\bar{\rho}_j - \rho_j)}
\]

where $C_c(\omega_k)_j = \{\bar{\rho}_j$ if $(\omega_k)_j = \rho_j$ or $\rho_j\}$ otherwise.

\[
\sum_{k=1}^{2^N} \alpha_k(\rho) = 1 , \alpha_k(\rho) > 0
\]
LPV control design

Polytopic approach

Solve the LMIs at each vertex of the polytope formed by the extremum values of each varying parameter, with a common $K$ Lyapunov function.

$$C'(\rho) = \sum_{k=1}^{2^N} \alpha_k(\rho) \begin{bmatrix} A_c(\omega_k) & B_c(\omega_k) \\ C_c(\omega_k) & D_c(\omega_k) \end{bmatrix}$$
LPV/$\mathcal{H}_\infty$ control synthesis

Proposition - feasibility (brief) Scherer et al. (1997)

Solve the following problem at each vertices of the parametrized points (illustration with 2 parameters):

$$\gamma^* = \min_{\gamma} \gamma$$

s.t. (69)

\[
\begin{bmatrix}
A X + B_2 \tilde{C}(\rho_1, \rho_2) + (\ast)^T \\
\tilde{A}(\rho_1, \rho_2) + A^T \\
B_1^T \\
C_1 X + D_{12} \tilde{C}(\rho_1, \rho_2)
\end{bmatrix}
\begin{bmatrix}
X \\
I \\
Y
\end{bmatrix}
\succ 0
\]

O. Sename [GIPSA-lab]
LPV/$\mathcal{H}_\infty$ control synthesis

Proposition - reconstruction (brief) Scherer et al. (1997)

Reconstruct the controllers as,

\[
\begin{align*}
C_c(\rho_1, \rho_2) &= \tilde{C}(\rho_1, \rho_2)M^{-T} \\
B_c(\rho_1, \rho_2) &= N^{-1}\tilde{B}(\rho_1, \rho_2) \\
A_c(\rho_1, \rho_2) &= N^{-1}(\tilde{A}(\rho_1, \rho_2) - YAX - NB_c(\rho_1, \rho_2)C_2X - YB_2C_c(\rho_1, \rho_2)M^T)M^{-T}
\end{align*}
\]

where $M$ and $N$ are defined such that $MN^T = I - XY$ which may be chosen by applying a singular value decomposition and a Cholesky factorization.
Definition LPV observers

Definition

Let consider the LPV system:

\[
\begin{align*}
\dot{x}(t) &= A(\rho)x(t) + B(\rho)u(t) \\
y(t) &= C(\rho)x(t)
\end{align*}
\] (72)

The following LPV state space representation

\[
\begin{align*}
\dot{\hat{x}}(t) &= A(\rho)\hat{x}(t) + B(\rho)u(t) + L(\rho)(y(t) - C(\rho)\hat{x}(t)) \\
\hat{x}_0 \text{ to be defined}
\end{align*}
\] (73)

is said to be an observer for (72) if

\[
\lim_{t \to \infty} (\hat{x}(t) - x(t)) \to 0 \quad \forall \rho(t) \in \Omega
\]

where \( \hat{x}(t) \in \mathbb{R}^n \) is the estimated state of \( x(t) \) and \( L(\rho) \) is the \( n \times p \) observer gain matrix to be designed.
Some issues for LPV observer design

The estimated error, $e(t) := x(t) - \hat{x}(t)$, satisfies:

$$\dot{e}(t) = (A - LC'(\rho)) e(t)$$  \hspace{1cm} (74)

The two main problems to be handle are then

- What observability property shall we consider?
- What parameter dependency should we define for $L(\rho)$?

Quadratic detectability (Wu, 95)

A simple solution is to consider a single Lyapunov function in order to guarantte the quadratic detectability, i.e:

$$\begin{align*}
(A(\rho) - L(\rho)C(\rho))^T P + P (A(\rho) - L(\rho)C(\rho)) &< 0
\end{align*}$$

Some remarks:

- The previous problem can be solved using a polytopic approach only if $C(\rho) = C$, a constant matrix.
- If this is not solvable, one can try using Parameter dependent Lyapunov functions, but the coupling between $L(\rho)$ and $P(\rho)$ will lead to solved non affine LMIs (a polynomial or a gridding approach is then needed).
Some issues for LPV observer design (2)

On key issue in observer implementation concerns the knowledge of $\rho(t)$. While previously the result is valid if $\rho(t)$ is perfectly known, such a following observer description must be used if $\rho(t)$ is estimated:

$$\dot{\hat{x}}(t) = A(\hat{\rho})\hat{x}(t) + B(\hat{\rho})u(t) + L(\hat{\rho})(y(t) - C(\hat{\rho})\hat{x}(t))$$  \hspace{1cm} (75)

Denoting $\Delta A = A(\rho) - A(\hat{\rho})$, $\Delta B = B(\rho) - B(\hat{\rho})$, $\Delta C = C(\rho) - C(\hat{\rho})$, and $\Delta L = L(\rho) - L(\hat{\rho})$, this leads for the estimation error equation:

$$\dot{e}(t) = (A - LC)(\hat{\rho})e(t) + (\Delta A + L(\hat{\rho})\Delta C)x + \Delta Bu(t)$$  \hspace{1cm} (76)

If $C(\rho) = C$ and $B(\rho)$ are constant matrices, then we get the uncertain estimated error system

$$\dot{e}(t) = (A(\hat{\rho}) - L(\hat{\rho})C)e(t) + \Delta Ax(t)$$  \hspace{1cm} (77)

The stability analysis is indeed more involved due to the state vector $x$ (see (Daafouz et al, 2010) for the discrete-time case). Either $\Delta Ax(t)$ should be considered as a disturbance, or a state augmentation approach is to be used (which has to be done in closed-loop control).

Observer-based control

For control design in the latter case, the following state feedback should be used:

$$u(t) = -F(\hat{\rho})\hat{x}(t)$$
Interest of the LPV approach

LPV is a key tool to the control of complex systems.

*Some examples:*

Modelling of complex systems (non linear)

- Use of a quasi-LPV representation to include non linearities in a linear state space model (even delays)
- Transformation of constraints (e.g. saturation) into an 'external' parameter
- Modelling of LTV, hybrid (e.g. switching control)

*BUT:*

A q-LPV system is not equivalent to the non linear one:

- stability: $\rho = \rho(x(t), t)$ is assumed to be bounded... so are the state trajectories
- controllability: some non controllable modes of a non linear system may vanish according to the LPV representation
- observability: unobservability may occur for some specific parameter variations
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Some of works using LPV approaches - former PhD students

Gain-scheduled control
- Account for various operating conditions using a variable "equilibrium point": (Gauthier 2007)
- Control with real-time performance adaptation using parameter dependent weighting functions from endogenous or exogenous parameters (Poussot 2008, Do 2011)
- Control under computation constraints: $H_\infty$ variable sampling rate controller with sampling dependent performances (Robert 2007, Roche 2011, Robert et al., IEEE TCST 2010)

Coordination of several actuators for MIMO systems
- An LPV structure for control allocation Poussot et al. (CEP 2011)
- Selection of a specific parameter for the control activation (of each actuator) Poussot et al. (VSD 2011), Doumiati et al (EJC 2013), Fergani et al (IEEE TVT 2015)

Incorporate fault-(diagnosis, accommodation, tolerant control) properties
Some PhD students on robust and/or LPV control

- Waleed Nwesaty, "LPV/$H_\infty$ control design of on-board energy management systems for electrical vehicles", PhD GIPSA-lab, Université Grenoble Alpes, 2015.
- Soheib Fergani, "$H_\infty$/LPV robust MIMO control of vehicle dynamics", PhD, GIPSA-lab, Université Grenoble Alpes, 2014.
- Maria Rivas, "Modeling and Control of a Spark Ignited Engine for Euro 6 European Normative", PhD, GIPSA-lab / RENAULT, Grenoble INP, 2012.
- David Hernandez, "Robust control of hybrid electro-chemical generators", PhD, GIPSA-lab / G2Elab, Grenoble INP, 2011.
- Emilie Roche, "Commande Linéaire à Paramètres Variants discrète à échantillonnage variable : application à un sous-marin autonome", PhD, GIPSA-lab, Grenoble INP, 2011.
- Corentin Briat, "Robust control and observation of LPV time-delay systems", PhD, GIPSA-lab, INP Grenoble, 2008.
- Christophe Gauthier, "Commande multivariable de la pression d’injection dans un moteur Diesel Common Rail", PhD, LAG / DELPHI, Grenoble INP, 2007.
- Julien Brely, " Régulation multivariable de filières de production de fibre de verre", PhD, LAG / ST Gobain Vetrotex, Grenoble INP, 2003.
- Giampaolo Filardi, "Robust Control design strategies applied to a DVD-video player", PhD, LAG / ST Microelectronics, Grenoble INP, 2003.
8 LPV systems

Summary of LPV approach interests

P. Gahinet.
A linear matrix inequality approach to $\mathcal{H}_\infty$ control.

C. Scherer.
The riccati inequality and state-space $\mathcal{H}_\infty$-optimal control.

C. Scherer and S. Wieland.
*LMI in control (lecture support, DELFT University)*.
2004.

C. Scherer, P. Gahinet, and M. Chilali.
Multiobjective output-feedback control via LMI optimization.

C. Scherer, P. Gahinet, and M. Chilali.
Multiobjective output-feedback control via lmi optimization.

S. Skogestad and I. Postlethwaite.
*Multivariable Feedback Control. Analysis and Design.*
John Wiley and Sons, Chichester, 1996.

*Robust and Optimal Control.*
New Jersey, 1996.