The quantization error in the average consensus problem

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***Abstract*—This work studies how uniform quantization in communications affect the linear diffusion algorithm for average consensus problem. Starting from the well-known linear diffusion algorithm, we analyze a simple and effective adaptation which is able to preserve the average of states and to drive the system near to the consensus value. The error is estimated by a worst case analysis, which suggests that it can increase as the number of agents goes to infinity. The algorithm is also compared with the existing literature.**

I. INTRODUCTION

In the last years, we have noticed an increasing interest for studying control, estimation and algorithmic problems over networks. A common feature of these problems is the fact that there is a fundamental constraint on the flux of information: data are distributed among a large number of nodes communicating among themselves through some network communication scheme. A prototype of such problems is the so-called consensus problem.

Suppose we have a (directed) graph $\mathcal{G}$ with set of nodes $V = \{1, \dots, N\}$ and a real quantity x_i for every node $i \in V$. The average consensus problem consists of computing the average $x_A = N^{-1} \sum_i x_i$ in an iterative and distributed way, exchanging information among nodes exclusively along the available edges in $\mathcal{G}$. This problem appears in a number of different contexts since the early 80's (decentralized computation [20], load balancing [7]) and, more recently, has attracted much attention for possible applications to sensor networks (data fusion problems [21], clock synchronization [16]) and to coordinated control of mobile autonomous agents [13], [3], [2], [9].

Suppose now that the links between agents do not allow a perfect exchange of information, but are noiseless digital channels on which symbols are sent. This is to say that the agents can just exchange quantized values. The simplest way of performing such a quantization is to decompose the domain of possible values in bins of equal size, and assign a symbol to each bin. This is uniform quantization. Up to a suitable rescaling, we may assume that the agents can exchange their values rounded to the nearest integer.

The consensus problem under quantization transmission has first appeared in the final section of [21]. In principle, we could simply try to take the usual linear algorithm which yields average consensus and try to apply it with quantization transmission. However, this turns out not to be a good idea since in the new setting the algorithm leads in general to an approximate consensus which may be far though from the average. This was noted in [21] and in [4], [6], where a new algorithm was proposed and partially analyzed. In this note, namely in Section III, we show that it is able to drive the system to average consensus, up to a small error due to quantization. The question of the magnitude of such error, and of its dependence on the number of agents, is addressed, and an answer is given by a theoretical analysis and by simulations. A more complete study is being presented in [10]. Some remarks on the related literature are given in Section V.

A. Notations and recalls

Before proceeding we collect some definitions and notations which are used through the paper. The communications between agents are modeled by a directed graph $\mathcal{G} = (V, E)$. $V = \{1, \dots, N\}$ is the set of vertices and E is the set of (directed) edges, i.e. a subset of $V \times V$. If $(j, i) \in E$, it means that j can transmit information about its state to i. The *adjacency matrix* A of $\mathcal{G}$ is a $\{0, 1\}$-valued square matrix indexed by the elements in V defined by letting $A_{ij} = 1$ if and only if $(j, i) \in E$ and $j \neq i$. Define the *in-degree* of a vertex i as $\sum_j A_{ij}$ and the *out-degree* of a vertex j as $\sum_i A_{ij}$. A graph is said to be *undirected* (or *symmetric*) if $(i, j) \in E$ implies that $(j, i) \in E$. A graph is strongly connected if for any given pair of vertices (i, j) there exists a path which connects i to j. A path in $\mathcal{G}$ consists in a sequence of vertices $(i_1, i_2, \dots, i_r)$ such that $(i_j, i_{j+1}) \in E$ for every $j \in \{1, \dots, r - 1\}$.

A matrix M is said to be *nonnegative* if $M_{ij} \geq 0$ for all i and j, and is said to be *doubly stochastic* if it is nonnegative and the sums along each row and column are equal to 1. A matrix M is said to be *normal* if, denoting with the star the conjugate transpose, $M^*M = MM^*$. Given a nonnegative matrix $M \in \mathbb{R}^{N \times N}$, we can define an induced graph $\mathcal{G}_M$ by taking N nodes and putting an edge (j, i) in E if $M_{ij} > 0$. Given a graph $\mathcal{G}$ on V, M is said to be *adapted* or *compatible* with $\mathcal{G}$ if $\mathcal{G}_M \subset \mathcal{G}$. Given a matrix $M \in \mathbb{R}^{N \times N}$, let $\sigma(M)$ denote the set of eigenvalues of M and $\rho(M)$ the spectral radius of M: $\rho(M) = \max\{|\lambda| : \lambda \in \sigma(M)\}$. When the matrix is stochastic, it is also worth to define the *essential* spectral radius as $\rho_{ess}(M) = \max\{|\lambda| : \lambda \in \sigma(M) \setminus \{1\}\}$.

II. STATEMENT OF THE PROBLEM

Assume that we have a set of agents V and a graph $\mathcal{G}$ on V describing the available links among the agents. For each agent $i \in V$ we denote by $x_i(t)$ the estimation of the average of agent i at time t.

In discrete time, if we have ideal exchange of information, the typical approach is to set as dynamics the following equations

$$x_i(t + 1) = \sum_{j=1}^{N} P_{ij} x_j(t), \tag{1}$$

where $x_i(t) \in \mathbb{R}$ is the state of the i-th agent at the time t and P_{ij} are coefficients belonging to a doubly stochastic matrix $P \in \mathbb{R}^{N\times N}$. More compactly we can write

$$x(t+1) = Px(t),$$

where $x(t)$ is the column vector whose entries $x_i(t)$ represent the agents states.

It is well known in the literature [19], [3] that, if P is a doubly stochastic matrix with positive diagonal and with $\mathcal{G}_P$ strongly connected, then the algorithm with exchange of perfect information (1) solves the *average consensus problem*, namely

$$\lim_{t\to+\infty} x(t) = \frac{1}{N}\left(\sum_{i=1}^{N} x_i(0)\right)\mathbf{1}, \tag{2}$$

where $\mathbf{1}$ is the (column) vector of all ones.

Assumption 1: The matrix P is doubly stochastic and such that $P_{ii} > 0$, $i = 1,\dots,N$, and $\mathcal{G}_P$ is strongly connected.
This will be assumed in the sequel of the paper.

When communication is quantized, we propose that at each time step each agent i adjourn its state following

$$x_i(t+1) = x_i(t) \,+\, \sum_{j=1}^{N} P_{ij}\left[q(x_j(t)) - q(x_i(t))\right],$$

where P_{ij} are the entries of a matrix P compatible with $\mathcal{G}$, and where $q(\cdot)$ denotes rounding to the nearest integer. If we have a vector $x \in \mathbb{R}^N$, with a slight abuse of notation, we will use the notation $q(x) \in \mathbb{R}^N$ to denote the vector such that $q(x)_i = q(x_i)$. Hence more compactly we can write

$$x(t+1) = x(t) \,+\, (P-I)\,q(x(t)). \tag{3}$$

where $x(t)$ is the column vector whose entries $x_i(t)$ represent the agents states.

The algorithm (3) preserves the average of the initial conditions, that is, defining $x_a(t) = N^{-1}\mathbf{1}^* x(t)$, $x_a(t) = x_a(0)$, $\forall t \in N$. Indeed, by Assumption refAssMatrixP, $\mathbf{1}^*P = \mathbf{1}^*$, and we have that, for all $t > 0$, $x_a(t+1) = N^{-1}\mathbf{1}^*x(t+1) = N^{-1}\mathbf{1}^*x(t) + N^{-1}\mathbf{1}^*\,(P-I)\,q(x(t)) = N^{-1}\mathbf{1}^*x(t) = x_a(t)$.

It is clear that the algorithm (3), because of the quantization effects, is not expected to converge in the sense (2). What we can hope is for the agents to reach real estimates which are close to each other and close to the average $x_a(0)$. To measure this asymptotic disagreement, we introduce the following quantity $\Delta_i(t) := x_i(t) - x_a(0)$. Since the average is preserved, $\Delta_i(t) = x_i(t) - x_a(t)$, and this represents the distance, at time t, of the i-th agent from the average of the initials conditions. Let now $Y = I - N^{-1}\mathbf{1}\mathbf{1}^*$ and $\Delta(t) = [\Delta_1(t),\dots,\Delta_N(t)]^*$. Then, $\Delta(t) = Y\,x(t)$. We define the performance index $d(P, x(0)) = \limsup_{t\to\infty} \frac{1}{\sqrt{N}}||\Delta(t)||$. To avoid a dependence on the initial condition, we prefer to consider

$$d_\infty(P) = \sup_{x(0)} d(P, x(0)).$$

III. Worst case analysis

An exact analysis of the dynamics of system (3) is an hard task and can be done only in special cases [10]. In this section we undertake a worst case analysis which is very general and proves that the proposed method drives the agents to a neighborhood of consensus, providing a bound on its size.

We start by observing that (3) can be rewritten in the following way

$$x(t+1) = Px(t) + (P-I)(q(x(t)) - x(t)), \tag{4}$$

where $q(x(t)) - x(t)$ is such that $\|q(x(t)) - x(t)\|_\infty \le 1/2$. In order to carry out a worst-case analysis of (4), we introduce the following bounded error model

$$\begin{cases} x_w(t+1) = Px_w(t) + (I-P)e(t), & x_w(0) = x(0)\\ \Delta_w(t) = Yx_w(t), \end{cases} \tag{5}$$

where $e(t) \in \mathbb{R}^N$ is such that $\|e(t)\|_\infty \le 1/2$ for all $t \ge 0$ and where we recall that $Y = I - \frac{1}{N}\mathbf{1}\mathbf{1}^*$. Notice that in this case $e(t)$ is no more a quantization error, but instead represents an unknown bounded disturbance. Clearly, when $e(t) = x(t) - q(x(t))$ it turns out that $x_w(t) = x(t)$ and $\Delta_w(t) = \Delta(t)$ for all $t \ge 0$.

We define now a performance index for (5), considering the worst asymptotic disagreement, worst with respect to all the possible choices of the time sequence of the vectors $e(t)$. To be more precise, let us introduce $\mathcal{E}^\infty = \left\{\{e(\cdot)\}_{t=0}^{\infty} \,|\, \|e(t)\|_\infty \le \frac{1}{2},\ \forall\, t \ge 0\right\}$, namely the set of all the sequences of N-dimensional vectors having sup norm less than $1/2$. Then, for the system (5), we define

$$d_\infty^w(P, x_w(0)) = \sup_{\mathcal{E}^\infty}\limsup_{t\to\infty}\frac{1}{\sqrt{N}}||\Delta_w(t)||.$$

Note that $\lim_{t\to\infty} YP^t = 0$. This implies that the asymptotic behavior of $\Delta_w(t)$ is *independent of the initial condition* $x_w(0)$ and hence this is the case also for the quantity $d_\infty^w(P, x_w(0))$. Thus, from now on we will denote $d_\infty^w(P, x_w(0))$ simply by $d_\infty^w(P)$. As a preliminary remark, note that

$$d_\infty(P) \le d_\infty^w(P).$$

We start from the following result that provides a general bound for d_∞^w.

Proposition 1: Let P be a matrix satisfying Assumption 1. Then $\|PY\| < 1$ and

$$d_\infty^w(P) \le \frac{1}{1 - \|PY\|}. \tag{6}$$

Proof: We have that$\|PY\| = \sqrt{\rho\left((PY)^*PY\right)}$. Since $PY = YP$ and $Y^2 = Y$ we can write that $(PY)^*PY = P^*PY$. Notice that the fact that P satisfies Assumption 1 implies both that $(P^*P)_{ii} > 0$ and $\mathcal{G}_{P^*P}$ is strongly connected. Therefore we can write that $\sigma(P^*P) = \{1, \lambda_1, \dots, \lambda_{N-1}\}$, where $|\lambda_i| < 1,\ 1 \le i \le N-1$. Observe that $\sigma(P^*PY) =$

$\{\sigma(P^*P) - \{1\}\} \cup \{0\}$. Hence $\|PY\| < 1$.

Consider now $\Delta_w(t)$. From (3), by simple algebra we have

$$\begin{aligned}\Delta_w(t) &= YP^t x(0) + Y\sum_{s=0}^{t-1} P^s(I-P)e(t-s-1) \\ &= (PY)^t\Delta(0) + \sum_{s=0}^{t-1}(PY)^s(I-P)e(t-s-1),\end{aligned}$$

where in the last equality we have used again the facts that $PY = YP$ and that $Y^k = Y$ for all $k > 0$. Now we have that

$$\begin{aligned}\|\Delta_w(t)\| &= \|(PY)^t\Delta_w(0) + \sum_{s=0}^{t-1}(PY)^s(I-P)e(t-s-1)\| \\ &\leq \|(PY)^t\|\,\|\Delta_w(0)\| + \|I-P\|\sum_{s=0}^{t-1}\|(PY)\|^s\|e(t-s-1)\| \\ &= \|(PY)^t\|\,\|\Delta_w(0)\| + \sqrt{N}\frac{1-\|PY\|^t}{1-\|PY\|},\end{aligned}$$

where in the last inequality we used the facts that $\|I-P\| \leq 2$ and $\|e(t)\| \leq \sqrt{N}/2$ for all $t \geq 0$. By letting $t \to \infty$ we obtain (6). ■

Note that, if P is normal we have that $\|PY\| = \rho_{ess}(P)$ and hence (6) becomes

$$d_\infty^w(P) \leq \frac{1}{1-\rho_{ess}(P)}.$$

However, when P is a normal matrix the bound on $d_\infty^w(P)$ can be improved as stated in the next proposition.

Proposition 2: If P is normal, then

$$d_\infty^w(P) \leq \frac{1}{2}\sum_{s=0}^{\infty}\rho(P^s(I-P)). \tag{7}$$

Proof: Starting from the expression of $\Delta_w(t)$ provided along the proof of Proposition 1 we can write that

$$\begin{aligned}\|\Delta_w(t)\| &\leq \|(PY)^t\Delta_w(0)\| + \|\sum_{s=0}^{t-1}(PY)^s(I-P)e(t-s-1)\| \\ &\leq \|(PY)^t\Delta_w(0)\| + \frac{\sqrt{N}}{2}\|(PY)^s(I-P)\|.\end{aligned}$$

Since P is normal we have that $\|(PY)^s(I-P)\| = \rho((PY)^s(I-P)) = \rho(P^s(I-P))$. By letting $t \to \infty$ in the last inequality, we obtain (7). ■

It is worth noting that, from the sub-multiplicative inequality $\|(PY)^s(I-P)\| \leq \|PY\|^s\,\|I-P\|$, it follows immediately that $\frac{1}{2}\sum_{s=0}^{\infty}\rho(P^s(I-P)) \leq \frac{1}{1-\rho_{ess}(P)}$ which shows that the bound (6) is indeed an improvement of the bound (7).

In general it is quite hard to evaluate (7). We provide two results which permit us to approximate (7) under some mild assumptions. First a notational definition. Given $c \in \mathbb{C}$ and $r \in \mathbb{R}$ such that $r \geq 0$, we denote

$$B_{c,r} := \{z \in \mathbb{C} \,|\, \|z-c\| \leq r\},$$

the closed ball of complex numbers of radius r and centered in c.

Proposition 3: Let P be a normal matrix satisfying the Assumption 1. Let R be such that $0 < R < 1$ and $\sigma(P) \subseteq B_{1-R,R}$ and let $\bar{\rho} = \rho_{ess}(P)$ denote the essential spectral radius of P. Then

$$\sum_{s=0}^{\infty}\rho(P^s(I-P)) \leq \frac{1}{1-R} + \sqrt{\frac{8R}{(1-R)(1-\bar{\rho})}}. \tag{8}$$

Proof: Assume that $\sigma(P) = \{\lambda_0 = 1, \lambda_1, \ldots, \lambda_{N-1}\}$. We want to upper bound $\rho(P^s(I-P)) = \max_{k=1}^{N-1}|\lambda_k^s(1-\lambda_k)|$. In order to do so we consider the function $f : \mathbb{C} \to \mathbb{R}$ defined as $f(z) = z^s(1-z)$. Let us consider the closed balls $B_{1-R,R}$ and $B_{0,\bar{\rho}}$. By Gershgorin's Theorem, $\sigma(P) \subseteq B_{1-R,R}$, and by definition of essential spectral radius, $\sigma(P) \setminus \{1\} \subseteq B_{0,\bar{\rho}}$. Hence $\sigma(P) \setminus \{1\} \subseteq B_{0,\bar{\rho}} \cap B_{1-R,R}$. Let $A := B_{1-R,R} \cap B_{0,\bar{\rho}}$. Clearly $\max_{k=1}^{N-1}|\lambda_k^s(1-\lambda_k)| \leq \max_{z \in A}|f(z)|$. Since f is an analytic function and A is a compact set, from the Maximum Modulus Principle it follows that

$$\max_{k=1}^{N-1}|\lambda_k^s(1-\lambda_k)| \leq \max_{z \in \partial A}|f(z)|,$$

where ∂A denotes the boundary of A.

Consider now the curves $\gamma, \theta : [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = 1 - R + Re^{jt}$, and $\theta(t) = \bar{\rho}e^{jt}$, which represent, respectively, the boundaries of $B_{1-R,R}$ and of $B_{0,\bar{\rho}}$. In the following, since $|f(z)| = |f(z^*)|$, we will consider γ and θ only on the interval $[0, \pi]$. Define $\xi = \frac{1-2R+\bar{\rho}^2}{2(1-R)}$. Computing the intersection between γ and θ, we get $\partial A = \tilde{\gamma} \cup \tilde{\theta}$ where $\tilde{\gamma} := \{z = z_x + iz_y \in \gamma : z_x \leq \xi\}$ and $\tilde{\theta} := \{z = z_x + iz_y \in \theta : z_x \geq \xi\}$. We consider now $|f(z)|$ on $\tilde{\gamma}$. By direct calculations one can show that

$$|f(\gamma(t))|^2 = 2R^2(1-\cos t)\left[1 - 2R(1-R)(1-\cos t)\right]^s.$$

Now let $x = R\cos t + 1 - R$ and let $F(x) = 2R(1-x)\left[1 - 2(1-R)(1-x)\right]^s$. Remark that studying $|f(z)|^2$ on $\tilde{\gamma}$ is equivalent to studying F on $\mathcal{I} := [1-2R, \xi]$.

Define $x_M = 1 - \frac{1}{2(1-R)(s+1)}$. F is such that it reaches its maximum on $\mathcal{I}$ in $1-2R$ if $x_M \leq 1-2R$, in x_M if $1-2R \leq x_M \leq \xi$, and in ξ if $x_M \geq \xi$. We have that $x_M \leq 1-2R \Leftrightarrow s \leq \frac{(1-2R)^2}{4R(1-R)}$, $1-2R \leq x_M \leq \xi \Leftrightarrow \frac{(1-2R)^2}{4R(1-R)} < s < \frac{\bar{\rho}^2}{1-\bar{\rho}^2}$, $x_M \geq \xi \Leftrightarrow s \geq \frac{\bar{\rho}^2}{1-\bar{\rho}^2}$. Let $\bar{s} = \lfloor\frac{(1-2R)^2}{4R(1-R)}\rfloor$ and $s^* = \lfloor\frac{\bar{\rho}^2}{1-\bar{\rho}^2}\rfloor$. Then,

$$\max_{x\in\mathcal{I}} F(x) = \begin{cases} 4R^2\left(1-2R\right)^{2s} & \text{if} \quad s \leq \bar{s} \\ \frac{R}{1-R}\frac{s^s}{(s+1)^{s+1}} & \text{if} \quad \bar{s}+1 \leq s \leq s^* \\ \frac{R}{1-R}\bar{\rho}^{2s}\left(1-\bar{\rho}^2\right) & \text{if} \quad s \geq s^*+1. \end{cases}$$

Consider now $|f(z)|^2$ on $\tilde{\theta}$. By simple algebraic manipulations one can see that $|f(\theta(t))|^2 = \bar{\rho}^{2s}\left(1 + \bar{\rho}^2 - 2\bar{\rho}\cos t\right)$. Note that $|f(\theta(t))|^2$ is monotone increasing for $t \in [0, \pi]$ and hence it reaches its maximum on $\tilde{\theta}$ when $\cos t = \frac{1-2R+\bar{\rho}^2}{2(1-R)}$. Therefore we can conclude that

$$\max_{k=1}^{N-1}|\lambda_k^s(1-\lambda_k)| \leq \begin{cases} 2R\left|1-2R\right|^s & \text{if} \quad s \leq \bar{s} \\ \sqrt{\frac{R}{1-R}\frac{s^s}{(s+1)^{s+1}}} & \text{if} \quad \bar{s}+1 \leq s \leq s^* \\ \sqrt{\frac{R}{1-R}\bar{\rho}^{2s}\left(1-\bar{\rho}^2\right)} & \text{if} \quad s \geq s^*+1. \end{cases}$$

Notice now that $\sum_{s=0}^{\bar{s}} 2R\,|1-2R|^s \leq \frac{2R}{1-|1-2R|} \leq \frac{1}{1-R}$, and that $\sum_{s=s^*+1}^{\infty} \sqrt{\frac{R}{1-R}\bar{\rho}^{2s}\left(1-\bar{\rho}^2\right)} \leq \sqrt{\frac{R}{1-R}}\sqrt{1-\bar{\rho}^2}\sum_{s=0}^{\infty}\bar{\rho}^s = \sqrt{\frac{R(1-\bar{\rho}^2)}{(1-R)(1-\bar{\rho})^2}} \leq \sqrt{\frac{2R}{(1-R)(1-\bar{\rho})}}$. Moreover, since $\sum_{i=1}^{m}\sqrt{\frac{1}{i+1}} \leq 2\sqrt{m+1}$, we can argue that $\sum_{s=\bar{s}+1}^{s^*}\sqrt{\frac{R}{1-R}\frac{s^s}{(s+1)^{s+1}}} \leq \sqrt{\frac{R}{1-R}}\sum_{s=1}^{s^*}\sqrt{\frac{1}{2}}\sqrt{\frac{1}{1+s}} \leq \sqrt{\frac{4R}{2(1-R)(1-\bar{\rho}^2)}} \leq \sqrt{\frac{2R}{(1-R)(1-\bar{\rho})}}$. Putting together these three inequalities we obtain (8). ■

Example 1 (Direct circuit): Suppose we have a dynamics like

$$x_i(t+1) = x_i(t) + \frac{1}{2}[-q(x_i(t)) + q(x_{i+1}(t))],$$

where $i = 1, \ldots, N$, and the summation of indexes is intended $\bmod N$. Such dynamics is also called *pursuing*. It can be written in compact form (3) with a suitable matrix P whose eigenvalues are (details in [3]) $\lambda_h(P) = \frac{1}{2} + \frac{1}{2}\exp\left(\frac{2\pi}{N}h\right)$ for $h = 1, \ldots, N$. Consider the bound introduced in Proposition 1. Since $\rho_{ess}(P) = 1 - \frac{\pi^2}{2}\frac{1}{N^2} + o\left(\frac{1}{N^2}\right)$ and since such matrix is normal, we have that $\frac{1}{1-\|PY\|} = \frac{1}{1-\rho_{ess}(P)} = \Theta\left(N^2\right)$. Observe now that all the eigenvalues of P are inside the ball $B_{\frac{1}{2},\frac{1}{2}}$. Hence we obtain that $\frac{1}{1-R} + \sqrt{\frac{8R}{(1-R)(1-\bar{\rho})}} = \Theta(N)$. This means that the bound (8) improves the bound proposed in (6). It is worth noting that in this case it is possible to prove ([10]) that $d_\infty \leq \frac{1}{2}$: estimating d_∞ by d_∞^w is not tight in general!

If P is symmetric we can provide a stronger result.

Proposition 4: Let P be a symmetric stochastic matrix satisfying Assumption 1. Let R be such that $0 < R < 1$ and $\sigma(P) \subseteq B_{1-R,R}$ and let $\bar{\rho} = \rho_{ess}(P)$ denote the essential spectral radius of P. Then,

$$\sum_{s=0}^{+\infty} \rho\left(P^s(I-P)\right) \leq \frac{3}{2} + \frac{1}{1-R} + \frac{1}{2}\log\left(\frac{1}{1-\bar{\rho}}\right). \qquad (9)$$

The proof is similar to the previous one, and we skip it here. It can be found in [10].

Example 2 (Undirect circuit): We apply this result to a concrete example. Suppose we have a dynamics like

$$x_i(t+1) = x_i(t) + \frac{1}{3}[q(x_{i-1}(t)) - 2q(x_i(t)) + q(x_{i+1}(t))],$$

where $i = 1, \ldots, N$, and the summation of indexes is intended $\bmod N$. It is clear that this dynamics corresponds to agents arranged like in a necklace (ring topology), and that it can be written in compact form (3) with a suitable matrix P. It is possible to prove (details in [3]) that its eigenvalues are $\lambda_h = \frac{1}{3} + \frac{2}{3}\cos\left(\frac{2\pi}{N}h\right)$ for $h = 1, \ldots, N$. By considering the bound (6), since $\rho_{ess}(P) \cong 1 - \frac{4}{3}\frac{\pi^2}{N^2}$ (see [3]) we obtain that $\frac{1}{1-\|PY\|} = \frac{1}{1-\rho_{ess}} = \Theta(N^2)$. Observe that all the eigenvalues of P are greater than $-\frac{1}{3}$. Hence it results, letting $R = 2/3$, that $\frac{3}{2} + \frac{1}{1-R} + \frac{1}{2}\log\left(\frac{1}{1-\bar{\rho}}\right) = \Theta(\log N)$.

We shall remark that there are cases in which the bound given by (7-9) is a tight estimate of $d_\infty^w(P)$. Actually we show that this happens for a sequence of matrices adapted to hypercube graphs. Also in this case the bound scales as the logarithm of the number of agents.

Example 3: The n-hypercube graph is the graph obtained drawing the edges of a n-dimensional hypercube. It has $N = 2^n$ nodes which can be identified with the binary words of length n. Two nodes are neighbors if the corresponding binary words differ in only one component. Thus every node has degree n. A matrix P can be constructed, adapted to the hypercube, taking its adjacency matrix A and defining $P = \frac{1}{n+1}(A+I)$. For such matrix, we can prove the following result.

Theorem 5: Let P be as above. Then

$$d_\infty^w(P) = \frac{n}{2} = \frac{\log_2 N}{2}.$$

The proof is deferred to the appendix.

IV. Scaling properties

Theorem 5 implies that there exist sequences of matrices such that $d_\infty^w(P_N)$ diverges with N and the estimate (7)-(9) is tight. However, simulations on the hypercube, Figure 1, do not show for d_∞, that is for the true system (3), this poor scaling.

As a further example we consider the random geometric graph. The random geometric graph is a graph of great applicative interest, since it is commonly used to model wireless networks [11]. It is constructed by placing, uniformly at random, N nodes in the unit square, and joining them with edges whenever their distance is below a threshold $R = \Theta(\sqrt{\log N/N})$ for $N \to \infty$. Also in these simulations, Figure 2, performance scales nicely with N.

Hence, we conclude that the bounded error approach has been useful to prove that the algorithm gets near to consensus, but is too pessimistic in evaluating its performance. A more complete analysis of the algorithm, which copes with this difficulty, and establishes more precise convergence results, is going to appear in [10].

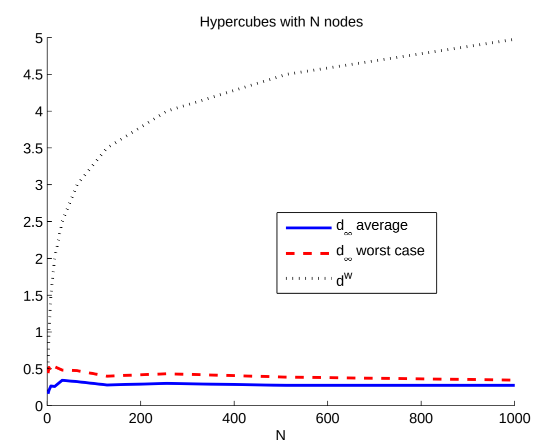

Fig. 1. The performance of the algorithm scales nicely on the hypercubes. We plot the average and the worst case of $d(P, x(0))$ over 100 trials, from simulations, and d_∞^w from theory.

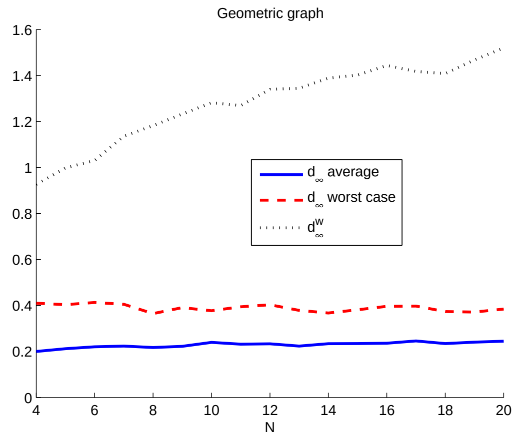

Fig. 2. The performance of the algorithm scales nicely on the random geometric graph. We plot the average and the worst case of $d(P, x(0))$ over 100 trials, and as d^w_∞ we plot the bound (7).

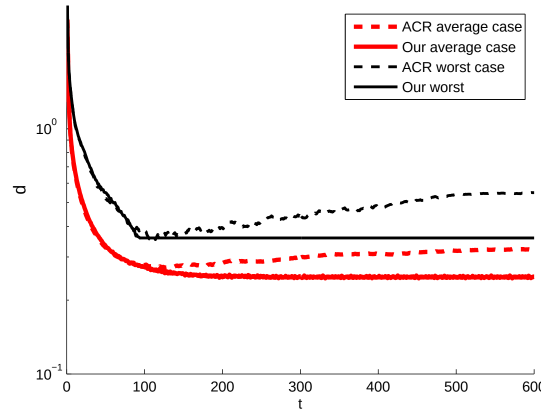

Fig. 3. Performance on a random geometric graph with $N = 50$ nodes, for the methods *ACR* [1], dashed lines, and (3), solid lines. Red lines represent the average of d, and black lines the worst case, with respect to the initial conditions.

V. This contribution in the literature

The effects of quantization in feedback control problems have been widely studied in the past [17], mainly in the stabilization problem. Notice moreover that granularity effects different from quantization in the consensus problems have been tackled in few papers, especially in the load balancing applications [8], [15].

In the last months, different quantized algorithms have appeared in the literature [15], [1], [18]. In [15], the authors study systems having integer-valued states and propose a class of gossiping[1] algorithms which preserve the average of states and are guaranteed to converge up to one quantization bin. The recently appeared technical report [18] studies a deterministic quantization scheme, not preserving the average, in which the agents can only store quantized data. Several results are obtained on the error between the convergence value and the initial average, and on the speed of convergence. The work [1] actually deals with quantized communication, and proposes to use a specific randomized quantizer together with a time invariant communication topology. This scheme achieves almost surely a consensus in the strict sense[2], and the consensus value is always one of the quantization levels, but it preserves the average only in expectation. Interestingly, in [1], the authors state for the expected squared distance from average consensus a bound which is very similar to (6). On the contrary, the algorithm (3), that we are studying in this note, preserves the average of initial conditions, and drives the system to a neighborhood of consensus.

Simulations of Figure 3 compare the two methods on a random geometric graph: one can see that the performance of (3) is slightly better.

[1]A gossip algorithm is a consensus algorithm in which agents exchange information in pairs, which are randomly selected. See [2] for a treatment without quantization.

[2]That is, all agents share the same state.

In some sense, using either our method, or that of probabilistic quantization [1], one trades off the precision in reaching the agreement among states and in preserving the average. Thus quantization seems to bring into the networked system some inaccuracy in estimating the average, that no know method is able to avoid. Are we facing an intrinsic limit of uniform quantization? A solution can be found in a more efficient use of the digital channel, as in [5], or in the time-varying strategy of [14], which allows to trade off the error from average and the speed of convergence.

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APPENDIX

The appendix is devoted to the proof of Theorem 5. Consider the hypercube graph and the matrix P as defined above. First we give the following preliminary result.

Lemma 6: Let P be as above. Then

$$\sum_{s=0}^{\infty} \rho(P^s(I-P)) = n = \log_2 N.$$

Proof: The eigenvalues of P are $\lambda_k = 1 - \frac{2k}{n+1} \quad k = 0 \dots n$, with multiplicities $p_k = \binom{n}{k}$.

Then, $\sum_{s=0}^{\infty} \rho(P^s(I-P)) = \sum_{s=0}^{\infty} \rho_{ess}(P^s)\rho(I-P) = \sum_{s=0}^{\infty}(1-\frac{2}{n+1})^s(2-\frac{2}{n+1}) = n.$ ■

We are able now to provide the proof of Theorem 5.

Proof: First we rewrite the expression of $d_\infty^w(P)$. Since P is symmetric, P is diagonalizable by an orthogonal matrix. We can write that $P = \sum_{h=0}^{N-1} \lambda_h q_h q_h^*$ where q_h are orthonormal. These facts are true also for $P^s(I-P)$. Moreover we have that $\rho\left(P^s(I-P)\right) = ||P^s(I-P)||$.

Let $\Delta_w^{(f)}(t) := \sum_{s=0}^{t-1} P^s(I-P)e(t-s-1)$. Then,

$$\begin{aligned}\|\Delta_w^{(f)}(t)\|^2 &= \|\sum_{s=0}^{t-1} P^s(I-P)e(t-s-1)\|^2 = \\ &= \|\sum_{s=0}^{t-1}\sum_{h=0}^{N-1} \lambda_h^s(1-\lambda_h)q_h q_h^* e(t-s-1)\|^2 \\ &= \|\sum_{h=0}^{N-1} q_h(1-\lambda_h)\sum_{s=0}^{t-1}\lambda_h^s q_h^* e(t-s-1)\|^2 = \\ &= \sum_{h=0}^{N-1}\left[(1-\lambda_h)\sum_{s=0}^{t-1}\lambda_h^s q_h^* e(t-s-1)\right]^2.\end{aligned}$$

Hence, $[d_\infty^w(P)]^2 = \max_{\mathcal{E}^\infty} \limsup_{t\to\infty} \frac{1}{N}\|\Delta_w^{(f)}(t)\|^2$.

Now we start using combinatorial tools. Indeed the vertices of the hypercube, as well as the eigenvalues and eigenvectors of P, can be indexed by the subsets of $\{1\,\dots,n\}$ (see [12]). With this indexing, for each $I \subseteq \{1\dots n\}$ the corresponding eigenvalue is $\lambda_I = 1 - \frac{2|I|}{n+1}$ and the eigenvector is the 2^n-dimesional vector $q^{(I)}$, such that its $J-$th component is equal to $q_J^{(I)} = 2^{-n/2}(-1)^{|I\cap J|}$.

Let T be any positive integer and consider the sequence of vectors $e(0), e(1), \dots, e(t), \dots$ such that J-th component of the vector $e(t)$ is equal to $\frac{1}{2}(-1)^{T-1-r}(-1)^{|J|}$, where r is the remainder in the euclidean division of t over T. Observe that $e(t+T) = e(t)$ for all $t \geq 0$. Observe, moreover, that $e(t)$ is an eigenvector of P corresponding to the eigenvalue $\frac{1-n}{1+n}$ for all $t \geq 0$. Hence we have that

$$\begin{aligned}\frac{1}{N}\|\Delta_w^{(f)}(T)\|^2 &= \frac{1}{N}\sum_{h=0}^{N-1}\left[(1-\lambda_h)\sum_{s=0}^{T-1}\lambda_h^s q_h^* e(T-s-1)\right]^2 = \\ &= \frac{1}{2^n}\left[\frac{2n}{n+1}\sum_{s=0}^{T-1}\left(\frac{1-n}{n+1}\right)^s \sum_J 2^{-\frac{n}{2}}(-1)^{|J|}\frac{1}{2}(-1)^s(-1)^{|J|}\right]^2 \\ &= \frac{1}{4^n}\left[\frac{n}{n+1}\sum_{s=0}^{T-1}\left(\frac{n-1}{n+1}\right)^s 2^n\right]^2 \\ &= \frac{n^2}{(n+1)^2}\left[\frac{1-\left(\frac{n-1}{n+1}\right)^T}{1-\frac{n-1}{n+1}}\right]^2 = \frac{n^2}{4}\left[1-\left(\frac{n-1}{n+1}\right)^T\right]^2.\end{aligned}$$

Assume now that T is an even positive integer. By recalling that $e(t+T) = e(t)$ for all $t \geq 0$, for $t = kT$ where $k \in \mathbb{N}$ it turns out that

$$\begin{aligned}\frac{1}{N}\|\Delta_w^{(f)}(kT)\|^2 &= \frac{n^2}{4}\left[1-\left(\frac{n-1}{n+1}\right)^T\right]^2\sum_{u=0}^{k-1}\left(\frac{1-n}{n+1}\right)^{uT} = \\ &= \frac{n^2}{4}\left[1-\left(\frac{n-1}{n+1}\right)^T\right]^2\left[\frac{1-\left(\frac{n-1}{n+1}\right)^{kT}}{1-\left(\frac{n-1}{n+1}\right)^T}\right] = \\ &\frac{n^2}{4}\left[1-\left(\frac{n-1}{n+1}\right)^{kT}\right]^2.\end{aligned}$$

Letting $k \to \infty$ we obtain that, for the particular sequence considered

$$\lim_{k\to\infty}\frac{1}{N}\|\Delta_w^{(f)}(kT)\|^2 = \frac{n^2}{4} \tag{10}$$

Therefore we have proved that

$$\limsup_{t\to\infty}\frac{1}{N}\|\Delta_w^{(f)}(kT)\|^2 \geq \frac{n^2}{4}$$

and hence $[d_\infty^w(P)]^2 \geq \frac{n^2}{4}$. Now, Lemma 6 implies that $[d_\infty^w(P)]^2 \leq \frac{n^2}{4}$, and then the claim follows. ■