EFFICIENT QUANTIZED TECHNIQUES FOR CONSENSUS ALGORITHMS

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Abstract: In the average consensus a set of linear systems has to be driven to the same final state which corresponds to the average of their initial states. This mathematical problem can be seen as the simplest example of coordination task. In fact it can be used to model both the control of multiple autonomous vehicles which all have to be driven to the centroid of the initial positions, and to model the decentralized estimation of a quantity from multiple measures coming from distributed sensors. This contribution presents a consensus strategy in which the systems can exchange information through a fixed strongly connected digital communication network. Beside the decentralized computational aspects induced by the choice of the communication network, here we also have to face the quantization effects due to the digital links. We present and discuss two different encoding/decoding strategies with theoretical and simulation results on their performance.

Keywords: Average Consensus, quantization, coding, coordination

1. INTRODUCTION

A basic aspect in the analysis and in the design of cooperative agents systems is related on the effect of the agents information exchange on the coordination performance. A coordination task which is widely treated in the literature is the so called average consensus. This is the problem of driving the states of a set of dynamic systems to a final common state which corresponds to the average of initial states of each system.

This mathematical problem can be shown to be relevant in the control of multiple autonomous vehicles which all have to be driven to the centroid of the initial positions, and in the decentralized estimation of a quantity from multiple measures coming from distributed sensors.

The way in which the information flow on the network influences the consensus performance has been already considered in the literature (Carli *et al.*, 2006a; Olfati *et al.*, 2004), when the communication cost is modelled simply by the number of active links in the network which admit the transmission of real numbers. However, this model can be too rough when the network links represent actual digital communication channels. Indeed the transmission over a finite alphabet requires the design of efficient ways to translates real numbers into digital information, namely smart quantization techniques. Some seminal research has been done on this problem (Kashyap *et al.*, 2006) (Carli *et al.*, 2007), even though the results are still quite preliminary due to the fact the quantizers are discontinuous nonlinear maps which, inserted into a dynamic system, produce complex behaviors.

While some efficient quantization methods have already been proposed (Carli *et al.*, 2006b), determining the optimal way to quantize data in this set-up is still a completely open problem. In this contribution we show how some quantization methods proposed in the field of control under communication constraints can be adapted to the average consensus problem. Namely, we will propose a first method based on a logarithmic quantizer (Elia *et al.*, 2001), and a second one, based on the so called *zooming-in/zooming-out* method (Brockett *et al.*, 2000; Tatikonda *et al.*, 2004; Nair *et al.*, 2005). In particular, this technique appears to be very promising for the design of efficient encoding/decoding communication schemes. For both of them, we present a theoretical convergence result, together with simulations.

2. PRELIMINARIES

Before defining the problem we want to solve, we summarize some notions on graph theory and we provide some notational conventions that will be useful throughout the rest of the paper.

Let $\mathcal{G} = (V, \mathcal{W})$ be a directed graph where $V = (1, \ldots, N)$ is the set of vertices and $\mathcal{W} \subset V \times V$ is the set of arcs. If $(i, j) \in \mathcal{W}$ we say that the arc (i, j) is outgoing from i and incoming in j. In our setup we admit the presence of self-loops. A path in $\mathcal{G}$ consists of a sequence of vertices $i_1 i_2 \ldots\ldots i_r$ such that $(i_\ell, i_{\ell+1}) \in \mathcal{W}$ for every $\ell = 1, \ldots, r-1$; i_1 (resp. i_r) is said to be the initial (resp. terminal) vertex of the path. A vertex i is said to be connected to a vertex j if there exists a path with initial vertex i and terminal vertex j. A directed graph is said to be connected if, given any pair of vertices i and j, either i is connected to j or j is connected to i. A directed graph is said to be strongly connected if, given any pair of vertices i and j, i is connected to j. A direct graph $\mathcal{G} = (V, \mathcal{W})$ is said to be a *circulant directed graph* if $(i, j) \in \mathcal{W}$ implies that $(i+p, j+p) \in \mathcal{W}$ for any $p \in \mathbb{N}$, where the sum is meant mod N. A graph is said to be undirected if $(i, j) \in \mathcal{W}$ implies that also $(j, i) \in \mathcal{W}$.

Now some notational conventions. Given a matrix $M \in \mathbb{R}^{N \times N}$, $diag\{M\}$ means a diagonal matrix with the same diagonal elements of the matrix M. Given a vector $m \in \mathbb{R}^N$, $diag\{m\}$ means a diagonal matrix having the components of m as diagonal elements. Given a vector $x \in \mathbb{R}^N$ with $\|x\|$ and $\|x\|_\infty$ we denote respectively the euclidean norm and the sup-norm. Accordingly, given a matrix $M \in \mathbb{R}^{N \times N}$, with $\|M\|$ and $\|M\|_\infty$ we denote the induced euclidean norm and the induced sup-norm.

3. PROBLEM FORMULATION

Consider $N > 1$ identical systems whose dynamics are described by the following discrete time state equations

$$x_i(t+1) = x_i(t+1) + u_i(t+1) \qquad i = 1, \ldots, N$$

where $x_i \in \mathbb{R}$ is the state of the i-th system and $u_i \in \mathbb{R}$ is the control input. More compactly we can write

$$x(t+1) = x(t) + u(t) \tag{1}$$

where $x, u \in \mathbb{R}^N$. The goal is to design an input control u yielding the consensus of the states, namely a control such that all the x_i's become equal asymptotically, i.e.

$$\lim_{t \to \infty} x(t) = \alpha \mathbb{1} \tag{2}$$

where $\mathbb{1} := (1, \ldots, 1)^T$ and α is a scalar depending on $x(0)$. Moreover, we also require that $x(t) = x(0)$ for all $t \in \mathbb{N}$ if $x(0) = \lambda \mathbb{1}$. An interesting case that has been widely studied in literature (see (Olfati *et al.*, 2004; Carli *et al.*, 2006a)) corresponds to the case in which $u(t)$ is a static feedback function of $x(t)$

$$u(t) = Kx(t), \quad K \in \mathbb{R}^{N \times N} \tag{3}$$

In such case the system (1) is described by the following closed loop system

$$x(t+1) = (I + K)x(t) \,. \tag{4}$$

It is easy to see that the consensus problem for system (4) is solved if and only if the following three conditions hold:

(A) the only eigenvalue of $I+K$ on the unit circle is 1;
(B) the eigenvalue 1 has algebraic multiplicity one (namely it is a simple root of the characteristic polynomial of $I + K$) and $\mathbb{1}$ is its eigenvector;
(C) all the other eigenvalues are strictly inside the unit circle.

In the sequel we will restrict to matrices K such that $I + K$ is a nonnegative matrix, namely a matrix with all elements nonnegative. Condition (B) then says that $I + K$ is a stochastic matrix. Conditions (A) and (C) yield the asymptotic behavior

$$(I + K)^t \to \mathbb{1} v^T$$

where $v \in \mathbb{R}^N$ is the unique probability vector such that $v^T (I + K) = v^T$. This implies that

$$x(t) \to v^T x(0) \mathbb{1}.$$

In the special case when $v = N^{-1} \mathbb{1}$ we obtain that the consensus is achieved at the average of the initial conditions. In this case $I + K$ is said to be a doubly stochastic matrix and K a *average consensus controller*.

We observe that the use of control law as in Equation (3) implies the exchange of perfect information through the communication network. More precisely, the fact that the element in position i, j of the matrix K is different from zero, means that the system i needs to know exactly the state of the system j in order to compute its feedback action. This implies that the agent j-th must communicate his state x_j to the system i. A good description of the communication effort required by a specific feedback K is given by the directed graph $\mathcal{G}_K$ with set of vertices $\{1, \ldots, N\}$ in which there is an arc from j to i whenever in the feedback matrix K the element $K_{ij} \neq 0$. The graph $\mathcal{G}_K$ is said to be the *communication graph* associated with K. Conversely, given any directed graph $\mathcal{G}$ with set of vertices $\{1, \ldots, N\}$, a feedback K is said to be *compatible* with $\mathcal{G}$ if $\mathcal{G}_K$ is a subgraph of $\mathcal{G}$ (we will use the notation $\mathcal{G}_K \subseteq \mathcal{G}$). The average consensus problem is said to be solvable on a graph $\mathcal{G}$ if there exists a feedback K compatible with $\mathcal{G}$ solving the average consensus problem. The following result completely characterizes those graphs for which the average consensus problem is solvable.

Proposition 3.1. Let $\mathcal{G}$ be a directed graph and assume that $\mathcal{G}$ contains all loops (i, i). The following conditions are equivalent:

(A) The average consensus problem is solvable on $\mathcal{G}$.
(B) $\mathcal{G}$ is strongly connected.

Furthermore, if the above conditions are satisfied, any K such that $I + K$ is doubly stochastic and $\mathcal{G}_{I+K} = \mathcal{G}$, solves the average consensus problem.

Now in our setup we assume that the communication network is constituted only of digital links. This implies that the exchange of perfect information between the systems is not allowed. In fact, through a digital channel, the i-th agent can only send to the j-th agent symbolic data that will be used by the j-th agent to built at most an estimate of the i-th agent's state. Here we consider a control law which has the same form of (3) where, in place of the exact knowledge of the states of the systems, we substitute estimates calculated according to the symbols sent through the communication network. More precisely, first we assume we have a fixed strongly connected graph $\mathcal{G}$ and a matrix K such that $I + K$ is doubly stochastic satisfying (A), (B) and (C) and $\mathcal{G}_{I+K} = \mathcal{G}$. The control input u_i has then the following form

$$u_i = \sum_{j=1}^{N} K_{ij} \hat{x}_{ij} \,, \tag{5}$$

where $\hat{x}_{ij}$ is the estimate of the state x_j which has been built by the agent i.

Now we proceed to explain how the estimate $\hat{x}_{ij}$ is obtained; we follow the treatment in the survey (Nair *et al.*, 2005). Suppose that the j-th agent sends to the i-th agent, through a digital channel, at each time instant t, a symbol $s_{ij}(t)$ belonging to a finite or denumerable alphabet $\mathcal{S}_{ij}$, called the *transmission alphabet*. It is assumed that the channel is reliable, that is each symbol transmitted is received without error.In general, the structure of the coder by which the j-th agent produces the symbol to be sent to the i-th agent can be described by the following equations

$$\begin{cases} \xi_{ij}(t+1) = F_{ij}(\xi_{ij}(t), s_{ij}(t)) \\ s_{ij}(t) = Q_{ij}(\xi_{ij}(t), x_j(t)) \end{cases} \tag{6}$$

where $s_{ij}(t) \in \mathcal{S}_{ij}$, $\xi_{ij}(t) \in \Xi_{ij}$, $Q_{ij} : \Xi_{ij} \times \mathbb{R} \to \mathcal{S}_{ij}$, and $F_{ij} : \Xi \times \mathcal{S}_{ij} \to \Xi_{ij}$. The decoder, placed at the system i, coincides with the system

$$\begin{cases} \xi_{ij}(t+1) = F_{ij}(\xi_{ij}(t), s_{ij}(t)) \\ \hat{x}_{ij}(t) = H_{ij}(\xi_{ij}(t), s_{ij}(t)), \end{cases} \tag{7}$$

where $H_{ij} : \Xi_{ij} \times \mathcal{S}_{ij} \to \mathbb{R}$.
Notice that the set Ξ_{ij} serves as *state space* for the coder/decoder, whereas the maps F_{ij}, Q_{ij}, H_{ij} represent, respectively, the *coder/decoder dynamics*, the *quantizer function*, and the *decoder function*. Coder and decoder are jointly initialized at $\xi_{ij}(0) = \xi_0$.
In general, we may have different encoders at system j, according to the various systems the system j wants to send its data. For the sake of notational convenience, we assume however, in this paper, that system j uses the same encoder for all data transmissions. Thus, system j will send the same symbol $s_j(t) := s_{ij}(t)$ to all the other systems i which receive information from it. In this case all systems receiving data from j, will obtain the same estimate of x_j, namely we can define a single state estimate $\hat{x}_j := \hat{x}_{ij}$. In this way, by letting $F_j = F_{ij}, H_j = H_{ij}, Q_j = Q_{ij}$ and $\Xi_j = \Xi_{ij}$, the previous coder/decoder couple can be represented by the following state estimator with memory

$$\begin{cases} \xi_j(t+1) = F_j(\xi_j(t), s_j(t)) \\ s_j(t) = Q_j(\xi_j(t), x_j(t)) \\ \hat{x}_j(t) = H_j(\xi_j(t), s_j(t)) \end{cases} \tag{8}$$

Moreover (5) assumes the following form

$$u_i = \sum_{j=1}^{N} K_{ij} \hat{x}_j \,, \tag{9}$$

It is worth observing that (9) preserves the average of the state x at each instant time as stated in the following proposition.

Proposition 3.2. Consider the system (1) and let $u(t) = [u_1(t), \ldots, u_N(t)]^T$ where $u_i(t)$ is of the form (5) for all i. Then

$$\mathbb{1}^T x(t+1) = \mathbb{1}^T x(t).$$

Proof 3.3. Notice that the control input can be rewritten in a vector form as $u(t) = K\hat{x}(t)$. Then

$$\begin{aligned}\mathbb{1}^T x(t+1) &= \mathbb{1}^T x(t) + \mathbb{1}^T K\hat{x}(t) \\ &= \mathbb{1}^T x(t)\end{aligned}$$

where in the last equality we have used the fact that $\mathbb{1}^T K = 0$.

An immediate consequence of the above proposition is that if u is a control input yielding the consensus of (1) then it yields the average consensus. The main objective of the present paper is to understand whether it is possible to design some smart encoding/decoding strategies such that a control law of the form (9) yields the average consensus for the overall system. In the sequel we concentrate our attention on two particular ways of exchanging information which fit into the previous scheme: the logarithmic quantized strategy and the zoom in-zoom out strategy.

4. LOGARITHMIC QUANTIZERS

This strategy is based on the techniques proposed in (Elia *et al.*, 2001). For $\delta \in]0,1[$, define the *logaritmic set of quantization levels*

$$\mathcal{S}_\delta = \left\{ \left(\frac{1+\delta}{1-\delta}\right)^\ell \right\} \bigcup \{0\} \bigcup \left\{ -\left(\frac{1+\delta}{1-\delta}\right)^\ell \right\}$$

The corresponding *logarithmic quantizer* $q^{(\delta)} : \mathbb{R} \to \mathcal{S}_\delta$ works as follows. Suppose that $x \in \mathbb{R}^+$ and let $i \in \mathbb{Z}$ be such that $\frac{(1+\delta)^{i-1}}{(1-\delta)^i} \leq x \leq \frac{(1+\delta)^i}{(1-\delta)^{i+1}}$, then define

$$q^{(\delta)}(x) = \left(\frac{1+\delta}{1-\delta}\right)^i.$$

If $x < 0$, define $q^{(\delta)}(x) = -q^{(\delta)}(-x)$. Finally, if $x = 0$, then $q^{(\delta)}(x) = 0$. Smaller values of the parameter δ corresponds to more accurate logarithmic quantizers $q^{(\delta)}$. Suppose now that the j-th agent is transmitting information to the i-th agent. For $\delta_j \in]0,1[$, the *logarithmic coder/decoder* is defined by the space $\Xi_j = \mathbb{R}$, the initial state $\xi_j(0) = 0$, the alphabet $\mathcal{S}_j = \mathcal{S}_{\delta_j}$ and by the maps

$$\begin{cases} \xi_j(t+1) = \xi_j(t) + s_j(t) \\ s_j(t) = q_j^{(\delta_j)}(x_j(t) - \xi_j(t)) \\ \hat{x}_j(t) = \xi_j(t) + s_j(t) \end{cases} \tag{10}$$

The coder/decoder pair is analyzed as follows. One can observe that $\xi_j(t+1) = \hat{x}_j(t)$, that is, the coder/decoder state contains the estimate of $x_j(t)$. The transmitted message contain a quantized version of the estimate error $x_j - \xi_j$. The estimate $\hat{x}_j(t)$ satisfies the recursive relation

$$\hat{x}_j(t+1) = \hat{x}_j(t) + q_j^{(\delta_j)}(x_j(t+1) - \hat{x}_j(t)),$$

with initial condition $\hat{x}_j(0) = q_j^{(\delta)}(x_j(0))$ determined by $\xi_j(0) = 0$.
Finally define the function $r : \mathbb{R} \to \mathbb{R}$ by $r(y) = \frac{q^{(\delta)}(y)-y}{y}$ for $y \neq 0$ and $r(0) = 0$. Some elementary calculations show that $|r(y)| \leq \delta$ for all $y \in \mathbb{R}$. Accordingly, if we define the trajectory $\epsilon_j : \mathbb{N} \to [-\delta, +\delta]$ by

$$\epsilon_j(t) = r(x_j(t+1) - \hat{x}_j(t)), \tag{11}$$

then we obtain that

$$\hat{x}_j(t+1) = \hat{x}_j(t) + (1+\epsilon_j(t))(x_j(t+1) - \hat{x}_j(t)). \tag{12}$$

5. ZOOMING IN ZOOMING OUT STRATEGY

This second strategy is inspired by the quantized stabilization technique proposed in (Brockett*et al.*, 2000), which is called zooming in - zooming out strategy. In this case the information exchanged between the agents is quantized by scalar uniform quantizers which can be described as follows. For $m \in \mathbb{N}$ define *uniform set of quantization levels*

$$\mathcal{S}_m = \left\{ -1 + \frac{2\ell - 1}{m} | \ell \in \{1, \dots, m\} \right\}.$$

The corresponding *uniform quantizer* $q^{(m)} : \mathbb{R} \to \mathcal{S}_m$ works as follows. Let $x \in \mathbb{R}$ then

$$q^{(m)}(x) = -1 + \frac{2\ell - 1}{m}$$

if $\ell \in \{1, \dots, m\}$ satisfies $-1 + \frac{2(\ell-1)}{m} \leq x \leq -1 + \frac{2\ell}{m}$, otherwise $q^{(m)}(x) = 1$ if $x > 1$ and $q^{(m)}(x) = -1$ if $x < -1$.
Suppose now that the j-th agent is transmitting information to the i-th agent. For $m_j \in \mathbb{N}$, $k_{in} \in]0,1[$, $k_{out} \in]1, +\infty[$, the *zooming in- zooming out uniform coder/decoder* has the state space $\Xi_j = \mathbb{R} \times \mathbb{R}_{>0}$, the initial state $\xi(0) = (0, \xi_j^2(0))$, where $\xi_j^2(0)$ is a suitable positive real number, and the alphabet $\mathcal{S}_j = \mathcal{S}_{m_j}$. The coder/decoder state is written as $\xi_j = (\xi_j^1, \xi_j^2)$ and the coder/decoder dynamics are

$$\xi_j^1(t+1) = \xi_j^1(t) + \xi_j^2(t) s_j(t)$$

and

$$\xi_j^2(t+1) = \begin{cases} k_{in}\xi_j^2(t) & \text{if } |s_j(t)| < 1 \\ k_{out}\xi_j^2(t) & \text{if } |s_j(t)| = 1 \end{cases}$$

The quantizer and decoder functions are, respectively,

$$s_j(t) = q_j^{(m_j)}\left(\frac{x_j(t) - \xi_j^1(t)}{\xi_j^2(t)}\right), \tag{13}$$

and

$$\hat{x}_j(t) = \xi_j^1(t) + \xi_j^2(t) s_j(t). \tag{14}$$

The coder/decoder pair is analyzed as follows. One can observe that $\xi_j^1(t+1) = \hat{x}_j(t)$, that is the first component of the coder/decoder state contains the estimate of $x(t)$. The transmitted messages contain a quantized version of the estimate $x_j - \xi_1^j$ scaled by the factor ξ_2^j. Accordingly, the second component of the coder/decoder state ξ_j^2 is referred to as the *scaling factor*: it grows when $|x_j - \xi_j^1| > \xi_j^2$ ("zooming out step") and decreases when $|x_j - \xi_j^1| \leq \xi_j^2$ ("zooming in step").

6. CONVERGENCE ANALYSIS

In this section we provide two convergence results regarding the methods previously illustrated. Consider first the logarithmic strategy. We assume that all the agents use the same quantizer function, that is $\delta_j = \delta$ for all j. The following result holds.

Theorem 6.1. Consider the system (1) where u is of the form (9) and where the estimates are built according to (12). Let $Y = I - \frac{1}{N}\mathbb{1}\mathbb{1}^T$. Then, if

$$\delta \leq \frac{1 - \|(I+K)Y\|}{\|K\|^2 + \rho\|I-K\|} \tag{15}$$

we have that, for any initial condition $x(0) \in \mathbb{R}^N$,

$$\lim_{t\to+\infty} x(t) = \lim_{t\to} \hat{x}(t) = \alpha\mathbb{1}$$

where $\alpha = \frac{1}{N}\mathbb{1}^T x(0)$.

Proof 6.2. Let $\epsilon_j(t)$ be defined as in (11) and let $\mathcal{E}(t) = diag\{\epsilon_1(t), \dots, \epsilon_N(t)\}$. Moreover let $\hat{x}(t) = [\hat{x}_1(t), \dots, \hat{x}_N(t)]^T$. Then (12) can be rewritten as

$$\hat{x}(t+1) = \hat{x}(t) + (I + \mathcal{E}(t))(x(t+1) - \hat{x}(t)).$$

Let us now to introduce the following new variables: $e(t) = \hat{x}(t) - x(t)$ and $y(t) = Yx(t)$. Notice that if $e(t) \to 0$ and $y(t) \to 0$ then $x(t) \to \alpha\mathbb{1}$ where by Proposition 3.2 $\alpha = \frac{1}{N}\mathbb{1}^T x(0)$. By straightforward calculations one can show that $y(t)$ and $e(t)$ satisfy the following recursive equations

$$\begin{cases} y(t+1) = (I+K)Yy(t) + Ke(t) \\ e(t+1) = \mathcal{E}(t)(Ky(t) + (K-I)e(t)) \end{cases}$$

By applying the sub-multiplicative and the triangular inequalities to the above equations it follows that

$$\begin{cases} \|y(t+1)\| \leq \|(I+K)Y\|\|y(t)\| + \|K\|\|e(t)\| \\ \|e(t+1)\| \leq \delta\|K\|\|y(t)\| + \delta\|K-I\|\|e(t)\| \end{cases}$$

where we have used the fact that $\|\mathcal{E}(t)\| \leq \delta$ for all t. Define now the sequences $\bar{y}(t)$ and $\bar{e}(t)$ as follows. Let $\bar{y}(0) = \|y(0)\|$ and $\bar{e}(0) = \|e(0)\|$ and let

$$\begin{cases} \bar{y}(t+1) = \|(I+K)Y\|\bar{y}(t) + \|K\|\bar{e}(t) \\ \bar{e}(t+1) = \delta\|K\|\bar{y}(t) + \delta\|I-K\|\bar{e}(t) \end{cases} \tag{16}$$

By straightforward calculations one can check that (16) is stable if and only if

$$\delta \leq \frac{1 - \|(I+K)Y\|}{\|K\|^2 + \rho\|I-K\|} \tag{17}$$

Moreover, by induction it can be proved that $\|y(t)\| \leq \bar{y}(t)$ and $\|e(t)\| \leq \bar{e}(t)$ for all t. Therefore, if δ satisfies (17) we have that, also $y(t) \to 0$ and $e(t) \to 0$, thus ensuring that the average consensus is asymptotically reached.

Remark 6.3. Under the hypothesis made on K, that is it satisfies the conditions (A), (B), (C), it is possible to prove that $\|(I+K)Y\| < 1$. Hence the quantity in (15) is strictly greater than 0.

Consider now the zooming in/ zooming out strategy. Again we assume that all the agents use the same quantizer function, that is $m_j = m$ for all j. Moreover we assume also that all the coder/decoder pair have the same initial conditions for the estimates and for the scaling factors: more precisely we impose $\hat{x}_j(0) = 0$ and $\xi_j^2(0) = \xi_0$. We have the following result.

Theorem 6.4. Consider the system (1) where u is of the form (9) and where the estimates are built according to (14). Let $Y = I - \frac{1}{N}\mathbb{1}\mathbb{1}^T$ and let $\rho = \|(I+K)Y\|$. Suppose that $\rho < k_{in} < 1$, $k_{out} = 1/k_{in}$, $m \geq \frac{(4+3k_{in})\sqrt{N}}{k_{in}(k_{in}-\rho)}$ and that $\xi_0 > \frac{2(\rho+2)\|x(0)\|}{k_{in} - \frac{3\sqrt{N}}{m}}$. Then we have that, for any initial condition $x(0) \in \mathbb{R}^N$,

$$\lim_{t\to+\infty} x(t) = \lim_{t\to} \hat{x}(t) = \alpha\mathbb{1}$$

where $\alpha = \frac{1}{N}\mathbb{1}^T x(0)$.

Proof 6.5. Let $\xi(t) = diag\{\xi_1^2(t), \dots, \xi_N^2(t)\}$, that is $\xi(t)$ is a diagonal matrix where $(\xi)_{jj}(t)$ represents the scaling factor relative to quantizer function of the j-th agent at the instant time t. We want to prove that under the hypothesis of the theorem, we have only zooming-in steps, namely $\xi(t) = k_{in}^t \xi_0 I$ for all $t \geq 0$. In order to do so we have to prove that for each instant time t the relation $|x_i(t+1) - \hat{x}_i(t)| \leq k_{in}^t \xi_0$ holds true for any i. From now on, throughout the proof we use the following notational convention: given two vectors $z = [z_1, \dots, z_N]^T, y = [y_1, \dots, y_N]^T \in \mathbb{R}^N$, $|z| \leq |y|$ means that $|z_j| \leq |y_j|$ for all j.
Let us now to introduce the following variables $\tilde{y}(t) = Kx(t)$ and $e(t) = x(t) - \hat{x}(t)$. One can show that if $e(t) \to 0$ and $y(t) \to 0$ then $x(t) \to \alpha\mathbb{1}$ where by Proposition 3.2 $\alpha = \frac{1}{N}\mathbb{1}^T x(0)$. It is possible to see that $\tilde{y}$ satisfies the following recursive equation

$$\tilde{y}(t+1) = (I+K)\tilde{y}(t) - K^2 e(t) \tag{18}$$

and that the condition $|x(t+1) - \hat{x}(t)| \leq \xi(t)\mathbb{1}$ translates into the condition $|\tilde{y}(t) - (K-I)e(t)| \leq$

$\mathcal{E}(t)\mathbb{1}$. Moreover if we assume that this last condition holds at the instant time t then $\|e(t+1)\| \leq \frac{1}{m}\|\mathcal{E}(t)\mathbb{1}\|$.

Observe now that, since $\hat{x}(0) = 0$ we have a zooming step at $t = 0$ if and only if $\|x(0)\|_\infty \leq \xi_0$. The validity of this last condition follows directly from the hypothesis on ξ_0, m and k_{in} and by the following calculations

$$\begin{aligned}\|x(0)\| &\leq \frac{\left(k_{in} - \frac{3\sqrt{N}}{m}\right)\xi_0}{2(\rho+2)} \\ &\leq \frac{k_{in}(4+3\rho)}{2(\rho+2)(4+3k_{in})}\xi_0 \\ &\leq \xi_0\end{aligned}$$

We prove now by induction on t that the following relations

$$\|\tilde{y}(t)\| \leq k_{in}^t \xi_0 \tag{19}$$

and

$$|\tilde{y}(t) - (K-I)e(t)| \leq k_{in}^t \xi_0 \mathbb{1}, \tag{20}$$

hold true, for each time step t. Observe that (20) implies that

$$\xi(t) = k_{in}^t \xi_0 I.$$

Notice preliminarily that $\|\tilde{y}(0)\| \leq 2\|x(0)\|$, $\|e(0)\| = \|x(0)\|$, $\|(I+K)\tilde{y}(t)\| = \|(I+K)Y\tilde{y}(t)\|$, $\|K^2\| \leq 4$ and $\|K-I\| \leq 3$. Let now $t = 1$. We have that

$$\begin{aligned}&\|\tilde{y}(1) - (K-I)e(1)\| \leq \\ &\quad\leq \|\tilde{y}(1)\| + \|K-I\|\|e(1)\| \\ &\quad\leq \|(I+K)Y\|\|\tilde{y}(0)\| + \|K^2\|\|e(0)\| + \\ &\qquad + \|K-I\|\|\mathcal{E}(0)\frac{1}{m}\mathbb{1}\| \\ &\quad\leq 2\rho\|x(0)\| + 4\|x(0)\| + 3\sqrt{N}\frac{\xi_0}{m} \\ &\quad= 2(\rho+2)\|x(0)\| + 3\sqrt{N}\frac{\xi_0}{m} \\ &\quad\leq \left(k_{in} - \frac{3\sqrt{N}}{m}\right)\xi_0 + \frac{3\sqrt{N}}{m}\xi_0 \\ &\quad= k_{in}\xi_0\end{aligned}$$

where in the last inequality we have used the fact that by hypothesis $2(\rho+2)\|x(0)\| \leq \xi_0\left(k_{in} - \frac{3\sqrt{N}}{m}\right)$. Since

$$\|\tilde{y}(1) - (K-I)e(1)\|_\infty \leq \|\tilde{y}(1) - (K-I)e(1)\|$$

we have that also

$$|\tilde{y}(1) - (K-I)e(1)| \leq k_{in}\xi_0\mathbb{1}.$$

Moreover

$$\begin{aligned}\|\tilde{y}(1)\| &\leq \|(I+K)Y\|\|\tilde{y}(0)\| + \|K^2\|\|e(0)\| \\ &\leq \rho\|\tilde{y}(0)\| + 4\|e(0)\| \\ &\leq 2\rho\|x(0)\| + 4\|x(0)\| \\ &\leq 2(\rho+2)\|x(0)\| \\ &\leq \xi_0\left(k_{in} - \frac{3\sqrt{N}}{m}\right) \\ &\leq k_{in}\xi_0\end{aligned}$$

Hence (19) and (20) hold for $t = 1$. Consider now a generic time step t and assume that (19) and (20) hold true for all the previous instants time and consider $\|\tilde{y}(t+1)\|$ and $\tilde{y}(t+1) - (K-I)e(t+1)$. We have that

$$\|\tilde{y}(t+1)\| \leq \rho\|\tilde{y}(t)\| + \|K\|^2\|e(t)\|,$$

and

$$\begin{aligned}&\|\tilde{y}(t+1) - (K-I)e(t+1)\| \leq \\ &\quad\leq \rho\|\tilde{y}(t)\| + \|K\|^2\|e(t)\| + \|K-I\|\|e(t+1)\|,\end{aligned}$$

By the inductive hypothesis it follows that

$$\|\tilde{y}(t)\| \leq k_{in}^t\xi_0,$$

$$\begin{aligned}\|e(t+1)\| &\leq \frac{1}{m}\|\xi(t)\mathbb{1}\| \\ &= \frac{1}{m}k_{in}^t\xi_0\sqrt{N}\end{aligned}$$

and

$$\begin{aligned}\|e(t)\| &\leq \frac{1}{m}\|\xi(t-1)\mathbb{1}\| \\ &= \frac{1}{m}k_{in}^{t-1}\xi_0\sqrt{N}\end{aligned}$$

Hence

$$\begin{aligned}\|\tilde{y}(t+1)\| &\leq \rho k_{in}^t\xi_0 + 4\frac{k_{in}^t(k_{in}-\rho)\xi_0}{4+3k_{in}} \\ &\leq \frac{3\rho+4}{3k_{in}+4}k_{in}^{t+1}\xi_0 \\ &\leq k_{in}^{t+1}\xi_0\end{aligned}$$

and

$$\begin{aligned}&\|\tilde{y}(t+1) - (K-I)e(t+1)\| \leq \\ &\quad\leq \rho k_{in}^t\xi_0 + \frac{1}{m}\left(\frac{4\sqrt{N}}{k_{in}} + 3\sqrt{N}\right)k_{in}^t\xi_0 \\ &\quad\leq \rho k_{in}^t\xi_0 + (k_{in}-\rho)k_{in}^t\xi_0 \\ &\quad= k_{in}^{t+1}\xi_0\end{aligned}$$

Again, since

$$\|\tilde{y}(t+1) - (K-I)e(t+1)\|_\infty \leq \|\tilde{y}(t+1) - (K-I)e(t+1)\|$$

we have that also

$$|\tilde{y}(t+1) - (K-I)e(t+1)| \leq k_{in}^{t+1}\xi_0 I.$$

This concludes the proof.

7. SIMULATION RESULTS

In this section we illustrate the behavior of the proposed algorithms by mean of simulations.

Namely we want to show how the performances of both methods depend on the parameters, and to compare them with the ideal communication case, in which the agents are able to communicate their state to their neighbors with a perfect communication, instead of a digital channel.

The comparison is done in two cases. We consider first a very special case, when the communication graph is a directed circuit and, i. e. a circulant directed graph with $(i, i+1) \in \mathcal{W}$ and $(i, i+r) \notin \mathcal{W}\ \forall r \neq 1$. Then we choose $K_{ij} = \frac{1}{2}\delta_{i,j} + \frac{1}{2}\delta i, j+1$, with δ the usual Kronecker delta.

Anyway, things are more interesting if we do not choose such a graph with special symmetries, but we take a graph closer to real problems. Since we are interested in decentralized estimation, we think to a wireless network, that we model as a random geometric graph. Random geometric graphs, with parameters N, R, are constructed by dropping N points randomly uniformly into the unit square and adding edges to connect any two points distant at most R from each other.

For both kinds of graphs, simulations of the logarithmic quantizer method (Figure 2) show that the method is convergent and that, increasing the parameter δ, its speed of convergence decreases.

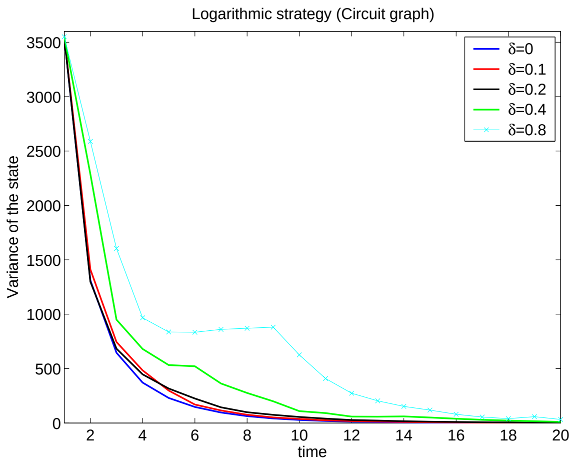

Fig. 1. Performance (variance of states) for the logarithmic quantizer method on a directed circuit of 20 agents for different values of δ.

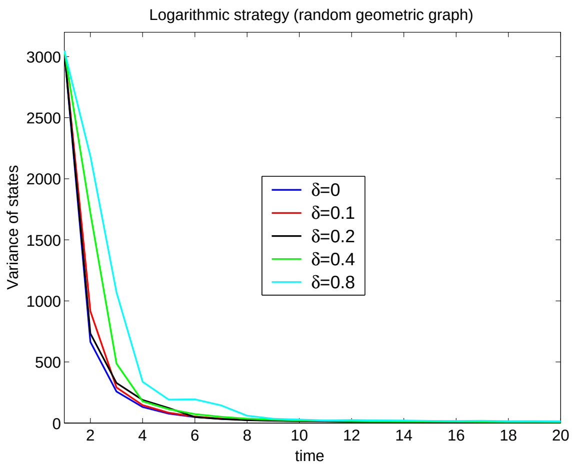

Fig. 2. Performance (variance of states) for the logarithmic quantizer method on a directed circuit of 20 agents for different values of δ.

For the zooming in-zooming out method, simulations are remarkable since they show that the it can be effective in many interesting cases. In the

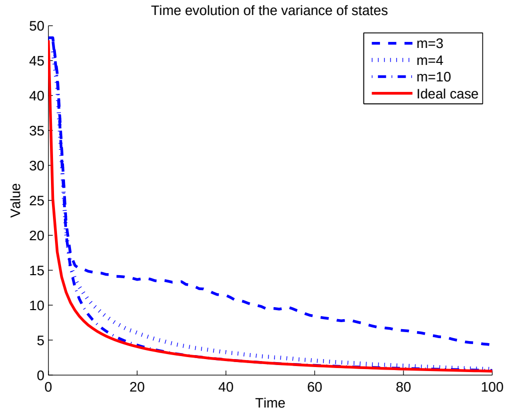

Fig. 3. Performance (variance of states) for the zooming in- zooming out method on a directed circuit of 20 agents for different values of m. $k_{in} = 0.9$, $k_{out} = 2$.

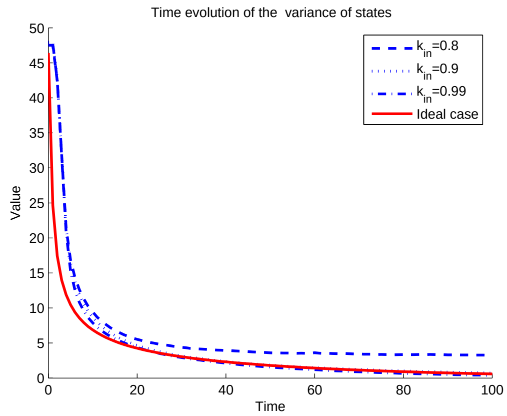

Fig. 4. Performance (variance of states) for the zooming in- zooming out method on a directed circuit of 20 agents for different values of k_{in}. $m = 10$, $k_{out} = 2$.

circuit case (Figures 4 and 3 [1]), a careful choice of the parameters allows to obtain very good performances in terms of the speed of convergence to the agreement, which is comparable to the ideal case.

Also in the geometric case the method performs very well, also in comparison with the ideal case, since it converges for a wide range of parameter choices, and moreover its speed of convergence seems to depend very slightly on the parameters, as can be seen in figures 6 and 5. [2]

8. CONCLUSIONS

In this paper we presented a new approach to the average consensus problem, where we considered

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[1] For these simulations we averaged over 200 realizations of $N = 20$ agents whose initial state were normally distributed with zero mean and variance $\sigma^2 = 50$.

[2] For these simulations we averaged over 1000 realizations of random geometric graph with radius $R = 0.3$ in the unit square with $N = 20$ agents, whose initial state were normally distributed as in the previous case.

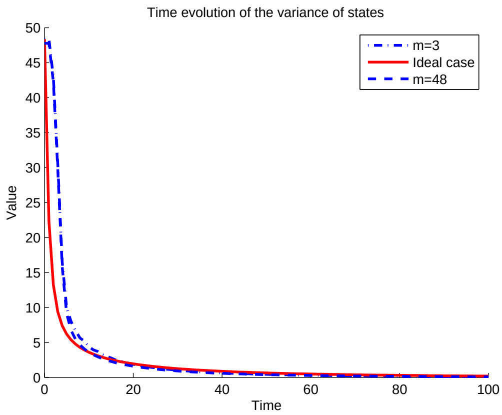

Fig. 5. Performance (variance of states) for the zooming in- zooming out method on a random geometric graph of 20 agents for different values of m. The convergence is almost not dependent on m. $k_{in} = 0.8$, $k_{out} = 2$.

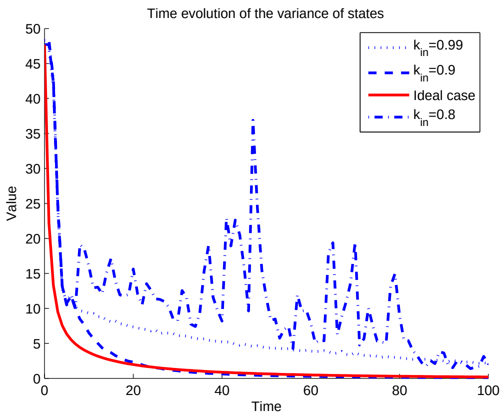

Fig. 6. Performance (variance of states) for the zooming in- zooming out method on a directed circuit of 20 agents for different values of k_{in}. $m = 4$, $k_{out} = 4$.

only quantized exchanges of information. In particular we considered two strategies, one based on logarithmic quantizers, and the other one based on a zooming in-zooming out strategy. We studied them with theoretical and experimental results proving that using these schemes the average consensus problem can be efficiently solved even if the agents can share only quantized information. Though the theoretical results are still quite conservative the efficiency of these methods is evident from simulations. Providing a more detailed theoretical analysis and extending these techniques to other motion coordination algorithms, like rendezvous in arbitrary dimension, deployment and cyclic pursuit, will be the object of our future investigations. An another field of future research will be to look for encoding and decoding methods which are able to solve the average problem also with noisy digital channels.

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