Robust self-triggered coordination by ternary controllers

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Coordination problem: consensus

The simplest and best known example of coordination:

- Consider n systems (integrators)

$$\dot{x}_i = u_i \qquad i \in I := \{1, \ldots, n\}$$

  linked by an undirected connected graph $G = (I, E)$.
  $\mathcal{N}_i$ is the set of neighbors of system i

- **Control problem:** Design inputs u_i, $i \in I$,
    - which depend on x_i and $\{x_j : j \in \mathcal{N}_i\}$ (local information),
    - such that

$$x_i - x_j \to 0 \quad \forall i, j$$

Why (still) studying consensus?

- It is a prototypical problem:
  solutions can help us to understand more complex problems
- It is useful in many application fields:
  - robotic networks
  - sensors networks
  - distribution networks
  - opinion dynamics
  - load balancing
- It is well studied:

Proposition (Standard consensus)

If the graph G is connected, the control law $u_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i)$

guarantees that $\lim_{t \to \infty} x_i(t) = c$ *for all* i*, where* $c = \sum_{j=1}^{n} \frac{x_j(0)}{n}$.

A constrained coordination problem

Standard consensus requires continuous flow of information from neighbors

this is too demanding!

We instead want a scenario in which

- sensors collect information only upon need $\longrightarrow$ discrete event times!
- the continuous-time systems "naturally" interacts with the discrete-time information acquisition
- the whole system is robust against network uncertainties (delays, poor synchronization of local clocks, limited data rate communication)

Hybrid system definition and main result

A hybrid coordination system I

State variables ($i \in I$)

- consensus variables: $x_i \in \mathbb{R}$
- control variables: $u_i \in \{-1, 0, +1\}$ (ternary controls)
- local clock variables: $\theta_i \in \mathbb{R}$

Continuous evolution when no information exchange occurs

$$\begin{cases} \dot{x}_i = u_i \\ \dot{u}_i = 0 \\ \dot{\theta}_i = -1 \end{cases}$$

Jumps occur at every t such that the set

$$\mathcal{I}(\theta, t) = \{i \in I \,:\, \theta_i = 0\} \neq \emptyset$$

A hybrid coordination system II

Discrete evolution: how the exchange of information affects the systems

$$\begin{cases} x_i(t^+) = x_i(t) \quad \forall i \in I \\ u_i(t^+) = \begin{cases} \mathrm{sign}_\varepsilon(\mathrm{ave}_i(t)) & \text{if } i \in \mathcal{I}(\theta, t) \\ u_i(t) & \text{otherwise} \end{cases} \\ \theta_i(t^+) = \begin{cases} f_i^\alpha(x(t)) & \text{if } i \in \mathcal{I}(\theta, t) \\ \theta_i(t) & \text{otherwise} \end{cases} \end{cases}$$

- $\mathrm{ave}_i(t) := \sum_{j \in \mathcal{N}_i} (x_j(t) - x_i(t))$ is the "consensus feedback"
- $\mathrm{sign}_\varepsilon(z) = \begin{cases} \mathrm{sign}(z) & \text{if } |z| \geq \varepsilon \\ 0 & \text{otherwise} \end{cases}$
- $\varepsilon > 0$ is a *sensitivity* parameter
- $\alpha \in (0, 1)$ is a *robustness* parameter

A hybrid coordination system III

Next sampling time is chosen by $\theta_i(t^+) = \begin{cases} f_i^\alpha(x(t)) & \text{if } i \in \mathcal{I}(\theta, t) \\ \theta_i(t) & \text{otherwise} \end{cases}$

$$f_i^\alpha(x) = \begin{cases} \dfrac{\alpha}{2\deg_i} \Big|\sum_{j\in\mathcal{N}_i}(x_j - x_i)\Big| & \text{if } \Big|\sum_{j\in\mathcal{N}_i}(x_j - x_i)\Big| \geq \varepsilon \\ \dfrac{\alpha}{2\deg_i}\varepsilon & \text{otherwise} \end{cases}$$

so that

- $\mathrm{sign}(\mathrm{ave}_i)$ is constant during inter-sampling interval (from t_k^i to t_{k+1}^i)
- **"dwell time"** property holds: $t_{k+1}^i - t_k^i \geq \dfrac{\alpha\varepsilon}{2\deg_{\max}}$
- $\varepsilon > 0$ is a *sensitivity* parameter
- $\alpha \in (0, 1)$ is a *robustness* parameter

Hybrid coordination protocol

Protocol A

initialization: for all $i \in I$, set $u_i(0) \in \{-1, 0, +1\}$ and $\theta_i(0) = 0$;
for all $i \in I$ **do**
    while $\theta_i(t) > 0$ **do**
        i applies the control $u_i(t)$;
    end while
    if $\theta_i(t) = 0$ **then**
        for all $j \in \mathcal{N}_i$ **do**
            i polls j and collects the information $x_j(t) - x_i(t)$;
        end for
        i computes $\mathrm{ave}_i(t)$;
        i computes $\theta_i(t^+) = f_i^\alpha(x(t))$;
        i computes $u_i(t^+) = \mathrm{sign}_\varepsilon(\mathrm{ave}_i(t))$;
    end if
end for

Main result

Theorem (Practical consensus)

For every initial condition $\bar{x}$, let $x(t)$ be the solution to Protocol A such that $x(0) = \bar{x}$. Then $x(t)$ converges in finite time to a point x^* belonging to the set

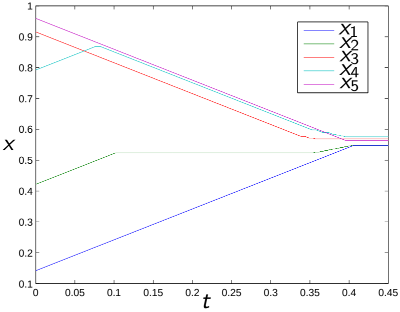
$$\mathcal{E} = \{x \in \mathbb{R}^n \, : \, |\sum_{j \in \mathcal{N}_i} (x_j - x_i)| < \varepsilon \; \forall \, i \in I\}$$

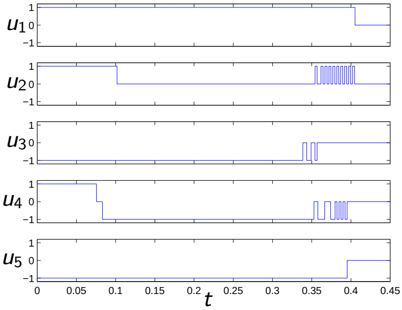
Proof sketch:

- Lyapunov-like argument $V(x) = \frac{1}{2} x^T L x = \frac{1}{2} \sum_{\{i,j\} \in E} (x_i - x_j)^2$
- we approximate the dynamics $\dot{x}_i = \mathrm{sign}(\mathrm{ave}_i)$, which is known to imply finite-time convergence

J. Cortés. Finite-time convergent gradient flows with applications to network consensus. *Automatica*, 42(11):1993–2000, 2006

Simulations





Sample evolutions of states x and corresponding controls u in Protocol A on a ring with $n = 5$ nodes, $\varepsilon = 0.02$

Robustness

Robustness: clock skews and quantized measurements

Continuous dynamics

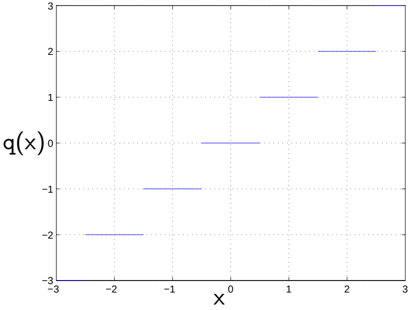
$$\begin{cases} \dot{x}_i = u_i \\ \dot{u}_i = 0 \\ \dot{\theta}_i = -\mathbf{R_i} \end{cases}$$

where $R_i > 0$ are local *(skewed)* clock rates

Quantized measurements: each system measures $\mathrm{q}(x_i - x_j)$

$$\mathrm{q}(x) = \Delta \left\lfloor \frac{x}{\Delta} + \frac{1}{2} \right\rfloor$$

$\Delta > 0$ is *accuracy* parameter



Discrete dynamics

$$\begin{cases} x_i(t^+) = x_i(t) \quad \forall i \in I \\ u_i(t^+) = \begin{cases} \mathrm{sign}_\varepsilon\left(\mathrm{qave}_i(t)\right) & \text{if } i \in \mathcal{I}(\theta, t) \\ u_i(t) & \text{otherwise} \end{cases} \\ \theta_i(t^+) = \begin{cases} f_i^\alpha(x(t)) & \text{if } i \in \mathcal{I}(\theta, t) \\ \theta_i(t) & \text{otherwise} \end{cases} \end{cases}$$

where

- $\mathrm{qave}_i(t) := \sum_{j \in \mathcal{N}_i} \mathrm{q}(x_j(t) - x_i(t))$
- $f_i^\alpha(x) = \begin{cases} \dfrac{\alpha}{2\deg_i} \left|\sum\limits_{j \in \mathcal{N}_i} \mathrm{qave}_i\right| & \text{if } \left|\sum\limits_{j \in \mathcal{N}_i} \mathrm{qave}_i\right| \geq \varepsilon \\ \dfrac{\alpha}{2\deg_i}\, \varepsilon & \text{otherwise} \end{cases}$

Robustness: clock skews and quantized measurements

Theorem (Clock skew & quantization robustness)

Assume that $R_i \geq R_{\min} > 0$ for all $i \in I$. If $\varepsilon > \frac{1}{2} d_{\max}\Delta$ and

$$\alpha < \frac{2\varepsilon - d_{\max}\Delta}{2\varepsilon} R_{\min},$$

then $x(t)$ converges in finite time to a point in

$$\mathcal{E}_2 = \{x \in \mathbb{R}^n \, : \, |\sum_{j \in \mathcal{N}_i} (x_j - x_i)| < 2\varepsilon\}$$

- Size of the region of convergence depends on quantizer resolution Δ
- $\alpha \in (0, R_{\min})$ quantifies the stability margin

Conclusion

Conclusion

Work done & its positive features:

- Coordination with self-triggered information collection (upon need)
- Coordination using coarse controllers and relative measurements
- Convergence of solutions with guaranteed dwell-time
- Finite-time convergence (with an explicit estimate)
- Other good properties in distributed systems:
  - No need for knowledge of absolute time
  - Robust against delays, quantization, clock skews, parameter uncertainties

Available extensions/variations (not presented here)

- Independent polling of neighbors & self-triggered gossiping
- Self-triggered protocols for asymptotical consensus ($x_i - x_j \to 0$)

Conclusion II

Future work

- Application to network flow control
- Extension to saturated controllers and other constrained controllers
- Extension to higher-dimensional systems
  (cf. joint work with J.M. Hendrickx)
- Extension to more complex coordination tasks (e.g., formation control of autonomous systems)

Details and related literature available in

C. De Persis and P. Frasca. Robust self-triggered coordination with ternary controllers. *IEEE Transactions on Automatic Control*, provisionally accepted. revised Dec. 2012

Thank you for your attention