Non-smooth and hybrid systems in opinion dynamics

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Outline

Basic opinion dynamics

Opinions $x_i(t) \in \mathbb{R}$ for population of individuals $i \in \mathcal{I} = \{1, \dots, N\}$

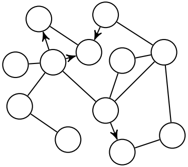
$$\dot{x}_i = \sum_{j=1}^{N} a_{ij}(x_j - x_i)$$

Opinions evolve through interactions between agents

- $a_{ij} = 1$ if j influences i; $a_{ij} = 0$ otherwise
- interactions described by the graph with adjacency matrix A

Additional notation:

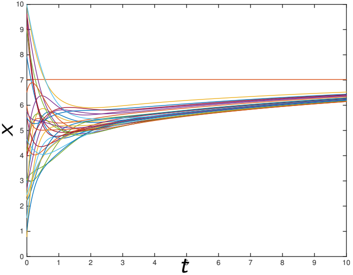
- degree $d_i = \sum_j a_{ij}$
- Laplacian $L = \text{diag}(d) - A$



Opinion dynamics *vs* consensus

If there is one node that can be reached from all other nodes
$\Longrightarrow$ convergence to consensus of opinions

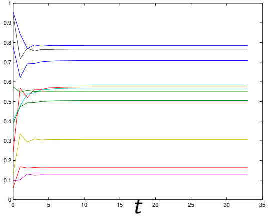
$$x_i(t) \to \alpha \in \mathbb{R} \text{ as } t \to +\infty \text{ for all } i \in \mathcal{I}$$



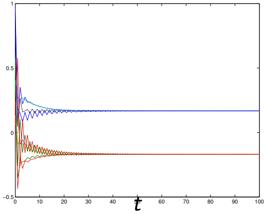
Issue: Societies do not exhibit consensus!

Models for disagreement: some potential causes

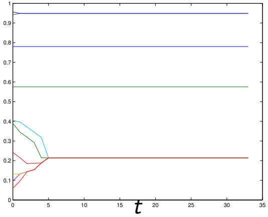
Prejudices



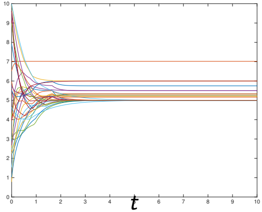
Antagonistic interactions



Bounded confidence



Discretized interactions



In this talk we focus on the last two $\Longrightarrow$ non-smooth systems

Non-smooth dynamical systems

Autonomous non-smooth systems

$\dot{x} = f(x)$ where $f : \mathbb{R}^N \to \mathbb{R}^N$ is discontinuous

Consequences:

- solutions are not smooth
- classical theorems fail to guarantee existence, uniqueness, completeness of solutions
- stability can be tricky (e.g. switching systems)

Well-studied topic, in books since [Clarke'83, Filippov'88]

Solutions for non-smooth systems

Let $I \subset \mathbb{R}$ be an interval of the form $(0, T)$.

- A continuously differentiable function $x : I \to \mathbb{R}^N$ is a classical solution if it satisfies $\dot{x} = f(x)$ for all $t \in I$
- An absolutely continuous function $x : I \to \mathbb{R}^N$ is a Carathéodory solution if it satisfies $\dot{x} = f(x)$ for almost all $t \in I$
  or, equivalently, if it is a solution of the integral equation

$$x(t) = x_0 + \int_0^t f(x(s))\mathrm{d}s$$

- An absolutely continuous function $x : I \to \mathbb{R}^N$ is a Krasovskii solution of $\dot{x} = f(x)$ if, for almost every $t \in I$, it satisfies

$$\dot{x}(t) \in \mathcal{K}f(x(t))$$

  where

$$\mathcal{K}f(x) = \bigcap_{\delta>0} \overline{\text{co}}(\{f(y) \ : \ y \text{ such that } \|x - y\| < \delta\})$$

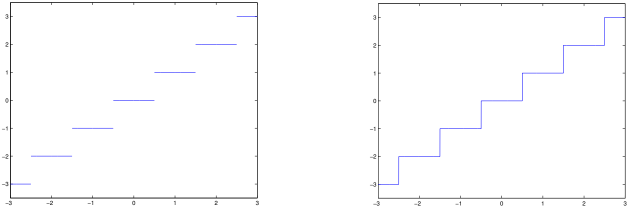
Krasovskii convexification

$$\mathcal{K}f(x) = \bigcap_{\delta>0} \overline{\text{co}}(\{f(y) \ : \ y \text{ such that } \|x - y\| < \delta\})$$

Examples:

- if f is continuous, then $\mathcal{K}f(x) = \{f(x)\}$
- if f has jumps, then $\mathcal{K}f(x)$ "fills" them

$f(x)$ $\qquad\qquad$ $\mathcal{K}f(x)$



Discrete behaviors

Quantized opinions and discrete behaviors

Behaviors are defined by a quantizer $q : \mathbb{R} \to \mathbb{Z}$ such that $q(s) = \lfloor s + \frac{1}{2} \rfloor$

$$\dot{x}_i = \sum_{j \in \mathcal{I}} a_{ij} \left(q(x_j) - x_i \right) \tag{Q}$$

Well-known in engineering. . .

Motivation: individuals are influenced by the others' behaviors [Friedkin'11]
limited verbalization [Urbig'03], discrete actions [Martins'08]

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Comparison with quantized consensus dynamics:

$$\dot{x}_i = \sum_{j \in \mathcal{I}} a_{ij} \left(q(x_j) - q(x_i) \right)$$ [Ceragioli,DePersis&F.'11]

$$\dot{x}_i = \sum_{j \in \mathcal{I}} a_{ij} \, q(x_j - x_i)$$ [Dimarogonas&Johansson'10]

these two dynamics approximately converge to consensus [Wei et al.'16]

Carathéodory solutions: good and bad news

Solutions to (Q)

From every initial condition there exists a complete Carathéodory solution

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Pathological attractors

It exists x^* such that $x(t) \to x^*$ but x^* is not equilibrium ?!

Example: $x(0) = (0, 0.49, 0.51, 1)$ on path graph

$$\begin{aligned}
\dot{x}_1 &= q(x_2) - x_1 = 0 \\
\dot{x}_2 &= q(x_1) + q(x_3) - 2x_2 = 1 - 2x_2 > 0 \\
\dot{x}_3 &= q(x_2) + q(x_4) - 2x_3 = 1 - 2x_3 < 0 \\
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\end{aligned}$$

asymptotically $x(t) \to x^* = (0, \frac{1}{2}, \frac{1}{2}, 1)$ but $f(x^*) = (1, 1, -1, -1)$

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asymptotically $x(t) \to x^* = (0, \frac{1}{2}, \frac{1}{2}, 1)$ but $f(x^*) = (1, 1, -1, -1)$

Krasovskii solutions avoid this pathology: if $x(t) \to \bar{x}$, then $0 \in \mathcal{K}f(\bar{x})$

Disagreement

On paths of length N:

$(0, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \dots, \frac{N-2}{2})$ is attractive but arbitrarily far from consensus

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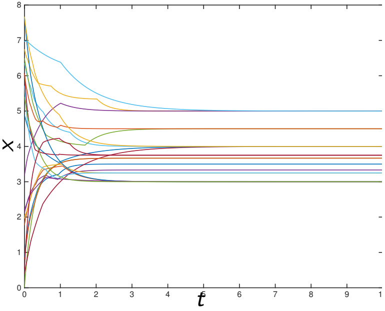
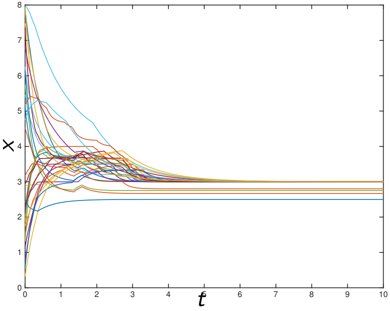
Lack of consensus is actually very common in simulations,
also on non-structured graphs:



Geometric random graph



Directed Erdős graph

Krasovskii solutions for large times

Asymptotical distance from consensus

Assume

- $x(t)$ is Krasovskii solution to (Q)
- the graph has symmetric adjacency matrix A
- $M = \left\{ x \in \mathbb{R}^N : \inf_{\alpha \in \mathbb{R}} \|x - \alpha \mathbf{1}\| \leq \frac{||A||}{\lambda_2} \frac{\sqrt{N}}{2} \right\}$

  λ_2 is smallest positive eigenvalue of L

then, $\mathrm{dist}(x(t), M) \to 0$ as $t \to +\infty$

Proof sketch:

- quantization error $x - q(x)$ is bounded
- Lyapunov function $V(x) = \frac{1}{2}||x - x_{\text{ave}}\mathbf{1}||_2^2$ with $x_{\text{ave}} := \frac{1}{N} \sum_{i=1}^{N} x_i$

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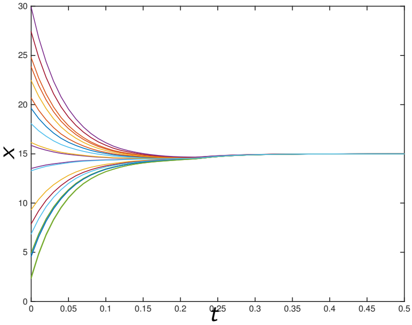
Note: M is tight: $\exists\, x^*$ such that $\frac{1}{\sqrt{N}}\|x^* - x^*_{\text{ave}}\mathbf{1}\| = \Theta(N^2)$ on path graphs

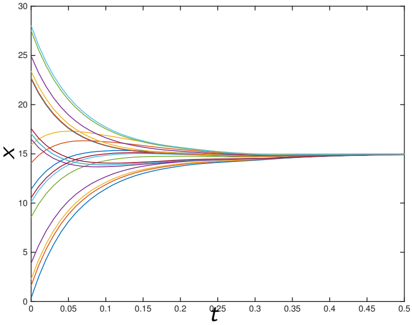
Consensus on special graphs

Special cases

Krasovskii solutions to (Q) converge to integer consensus $x^* = k\mathbf{1}$

- if the graph is complete; or
- if the graph is complete bipartite





Conclusions on discrete behaviors

Summary:

1. This was the simplest possible model. . .
2. Quantized behaviors can explain disagreement
3. Preferred notion of solutions is Krasovskii

. . . see [Ceragioli&F.,'15] and [Ceragioli&F.,'16]

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Open problems:

- Does dynamics (Q) converge?
- Are there closed solutions/limit cycles?
- Are there any non-Caratheodory non-constant solutions with non-negligible basin of attraction?
- Necessary and sufficient conditions for consensus (which topologies?)

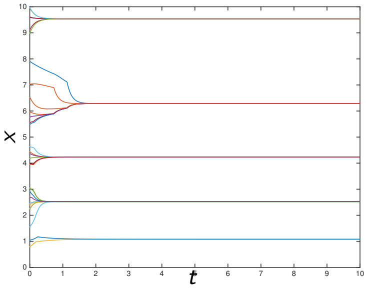
Bounded confidence

Bounded confidence

Model: Confidence threshold $R > 0$

$$\dot{x}_i = \sum_{j:|x_i - x_j| < R} (x_j - x_i) \qquad \text{(BC)}$$

motivated by [Hegselmann&Krause'02] and proposed by [Blondel et al.'10]



- Discontinuous right-hand side
  $a_{ij} = 1$ if $|x_i - x_j| < R$
- Formation of disconnected clusters where individuals agree

Existence of solutions

Solutions to (BC)

From almost every initial condition there exists a complete unique Carathéodory solution

From every initial condition there exists a complete Krasovskii solution

Carathéodory solutions $\subsetneq$ Krasovskii solutions

Example: $N = 3$, $R = 1$

$x(0) \in \{x \,:\, |x_1 - x_2| < 1, x_3 - x_2 = 1\}$

$$\dot{x} \in \left\{ \alpha \begin{bmatrix} x_2 - x_1 \\ 1 + x_1 - x_2 \\ -1 \end{bmatrix} + (1-\alpha) \begin{bmatrix} x_2 - x_1 \\ x_1 - x_2 \\ 0 \end{bmatrix} \,:\, \alpha \in [0,1] \right\}$$

which can be normal to the discontinuity surface

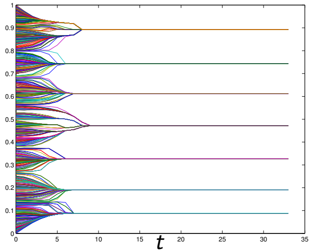
Equilibria and convergence

Krasovskii solutions to (BC)

1. Equilibria are $E = \{x \,:\, \text{for every } (i,j) \text{ either } x_i = x_j \text{ or } |x_i - x_j| \geq R\}$
2. $x_{\text{ave}}(t) = x_{\text{ave}}(0)$
3. $x(t) \to x^* \in E$ as $t \to +\infty$

Proof sketch:

- Order preservation
- Contractivity and boundedness
- Lyapunov function $V(x) = \frac{1}{2}x^\top x$
- Invariance Principle [Ceragioli'00]



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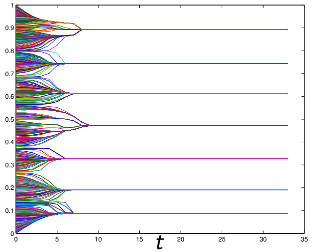

But E is not strongly invariant and is not stable

Example: Take $N = 2$ and $R = 1$ and the solution $x(t) = (\frac{1}{2} + \frac{1}{2}e^{-2t}, \frac{1}{2} - \frac{1}{2}e^{-2t})$ leaving from $x(0) = (1, 0)$ to $x(t) \to (\frac{1}{2}, \frac{1}{2})$

Robustness of equilibria

Definition: Equilibrium $x \in E$ is robust if no perturbation consisting in adding one agent causes two of the former clusters to coalesce in the resulting evolution

Let $x \in E$ and consider two clusters in x, denoted by A and B, having values x_A and x_B and cardinalities $n_A \leq n_B$

Robustness (R=1)

For the equilibrium $x \in E$ to be *robust* it is

- *sufficient* that $|x_B - x_A| > 2$ for every A, B
- *necessary* that $|x_B - x_A| > 1 + \frac{n_A}{n_B}$ for every A, B

For large N, $|x_B - x_A| > 1 + \frac{n_A}{n_B}$ becomes approximately sufficient, too

Conclusions on bounded confidence

Again,

1. This was the simplest possible model
2. Bounded confidence can explain disagreement
3. Preferred notion of solutions is Krasovskii

...see [Ceragioli&F.,'11] for a discussion on the role of discontinuities

Bounded confidence: a hybrid system

Hybrid Laplacian dynamics

Potential edges (i, j) have status of variables $a_{ij} \in \{0, 1\}$

$$\begin{cases} \dot{x}_i = \sum_{j \in \mathcal{I} \setminus \{i\}} a_{ij}(x_j - x_i) & \text{for all } i \in \mathcal{I} \\ \dot{a}_{ij} = 0 & \text{for all } (i, j) \in \mathcal{I} \times \mathcal{I} \end{cases} \qquad \text{(Flow)}$$

$$\begin{cases} x_i^+ = x_i \quad \text{for all } i \in \mathcal{I} \\ a_{hk}^+ = 1 - a_{hk} \\ a_{ij}^+ = a_{ij} \quad \text{for all } (i, j) \neq (h, k) \end{cases} \qquad (x, a) \in D_{hk} \qquad \text{(Jump)}$$

Jump set: $D = \bigcup_{hk} D_{hk}$

Flow set: $C = \overline{X \setminus D}$

Jump set

Bounded confidence with hysteresis regularization:

$$
\begin{aligned}
D_{hk}^{\text{on}} :=& \{a_{hk} = 0\} \cap \{(x_h - x_k)^2 \leq R^2 - \varepsilon\} \\
D_{hk}^{\text{off}} :=& \{a_{hk} = 1\} \cap \{(x_h - x_k)^2 \geq R^2 + \varepsilon\} \\
D_{hk} :=& D_{hk}^{\text{off}} \cup D_{hk}^{\text{on}}
\end{aligned}
$$

where R and ε are positive scalars and ε is (much) smaller that R

Remarks:

- ε-close approximation of the previous non-smooth model
- Well-posed and chattering-free dynamics

Convergence and stability properties

Let $\tilde{E} = \{(x, a) \,:\, a_{ij}(x_i - x_j) = 0 \text{ for all } (i, j)\}$ (i.e. $a_{ij} = 1 \Rightarrow x_i = x_j$)

Convergence of hybrid dynamics

- $a(t)$ has a finite number of jumps
- $(x(t), a(t)) \to (x^*, a^*) \in \tilde{E}$ as $t \to +\infty$
- (x^*, a^*) is such that $x_i^* = x_j^*$ if $a_{ij}^* = 1$
  and $|x_i^* - x_j^*| \geq R^2 - \varepsilon$ if $a_{ij}^* = 0$

Proof sketch:

- Boundedness
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- Invariance Principle [Goebel,Sanfelice&Teel'12]

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Proof sketch:

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Note: The set $\tilde{E}$ is not invariant and not stable:
take $(a, x) \in \tilde{E}$ such that $a_{ij} = 0$ and $x_i - x_j = R^2 - \varepsilon$

Conclusion

Summary

- Opinion dynamics can be written as hybrid dynamics
- General tools can be used to study their stability and convergence

please read the related work in [F.,Tarbouriech&Zaccarian'16]

Outlook

a. More complex jump rules
b. Combining quantization and bounded confidence

Conclusion

General conclusion

Summary

1. Opinion dynamics naturally lead to discontinuous/hybrid systems
2. Generalized solutions and Lyapunov theories are useful for analysis
3. Interesting and precise results can be obtained

   completeness, equilibria, convergence, robustness
4. Pathologies abound (mainly, convergence without stability)

Outlook

a. What to do these discontinuous/hybrid models?

b. What is the meaning for social sciences?

Works on which the talk is based (in collaboration)

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