Harmonic influence in social networks

Identification of influencers by message passing

Paolo Frasca

based on joint works with
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A. Ozdaglar (MIT)
W.S. Rossi (Twente)
L. Vassio (Torino)

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What is social influence?

What is the most influential node in a network?

Context-dependent question:
opinion dynamics // epidemic spread // cascading activation // resource competition // ...

D. Kempe, J. Kleinberg, and E. Tardos. Maximizing the spread of influence through a social network. *ACM SIGKDD '03*, 2003.

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In this talk:

A leader competes against an adversary field to influence the opinions of the other individuals

Y. Yildiz, A. Ozdaglar, D. Acemoglu, A. Saberi, and A. Scaglione. Binary opinion dynamics with stubborn agents. *ACM Transactions on Economics and Computation*, 2013

Our approach: harmonic influence

Which leader location (node) maximizes the influence?

1. We define the harmonic influence of a node
2. We relate social and electrical networks
3. We derive a message-passing algorithm
4. We prove its convergence
5. We discuss a few simulations

L. Vassio, F. Fagnani, P. Frasca, and A. Ozdaglar. Message passing optimization of harmonic influence centrality. *IEEE Transactions on Control of Network Systems*, 2014

W.S. Rossi and P. Frasca. The harmonic influence in social networks and its distributed computation by message passing, 2016, `http://arxiv.org/abs/1611.02955`

Influence Maximization

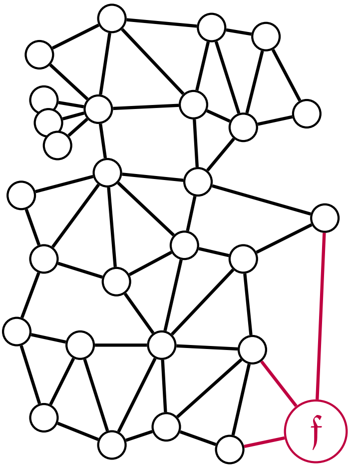
Opinions in the social network

Each individual i has **opinion** $x_i(t) \in \mathbb{R}$ evolving with time

Opinions evolve through

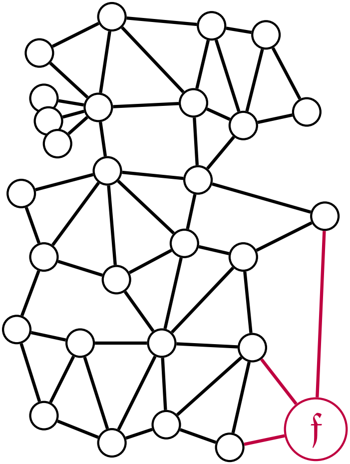
- *social interactions* between individuals
- influence of an external field

Weighted graph $\mathcal{G} = (I, E, C)$
- node set $I = \{\mathfrak{f}, 1, 2, \ldots, n\}$
- $\mathfrak{f}$ is a special *field* node
- undirected edge set E
- *non-negative weight matrix* C such that $C_{ij}C_{ji} > 0 \Leftrightarrow \{i, j\} \in E$



Opinions dynamics in the social network

We introduce a leader against the field



Opinions dynamics in the social network

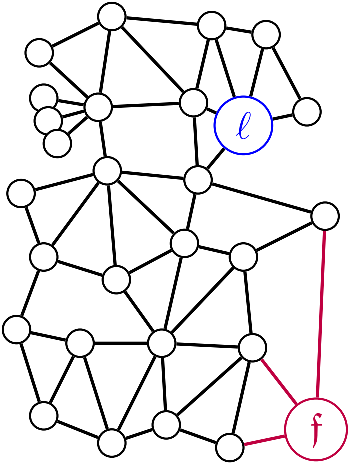
We introduce a leader against the field

- The field $\mathfrak{f}$ is stubborn

$$x_{\mathfrak{f}}(t) = x_{\mathfrak{f}}(0) \qquad \text{for all } t$$

- The **leader** ℓ is also stubborn

$$x_{\ell}(t) = x_{\ell}(0) \qquad \text{for all } t$$



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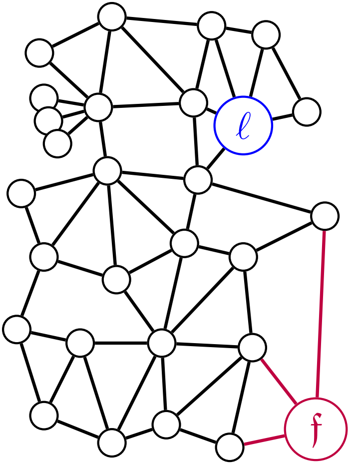
- The remaining **individuals** do local averaging

$$x_i(t+1) = \sum_{j \neq i} Q_{ij}\, x_j(t)$$

where $Q = D^{-1}C$
with diagonal matrix $D = \text{diag}(C\mathbf{1})$



Harmonic influence

Let *Laplacian* matrix $L = D - C$
Normalize opinions in $[0, 1]$

Dirichlet problem

Equilibrium opinions solve Laplacian system with boundary conditions

$$\begin{cases} L\mathbf{x} = \mathbf{0} \\ x_\ell = 1 \\ x_\mathfrak{f} = 0 \end{cases}$$

The *Harmonic Influence of ℓ* is $H(\ell) := \mathbf{1}^\top \mathbf{x}$
($\mathbf{x}$ is said to be a harmonic function)

Computing H requires solving n linear systems, one for each possible leader

Computing the Harmonic Influence

Problem:

Find an algorithm that

- solves all n systems at the same time
- is distributed

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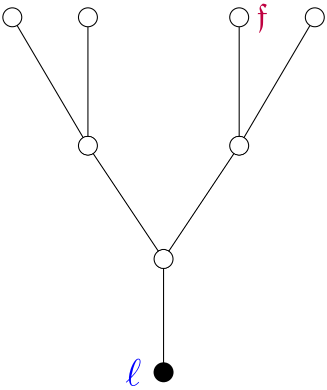
Solution:

Message-passing iterative algorithm that approximates H

- with provable convergence
- with insights on convergence speed and approximation error

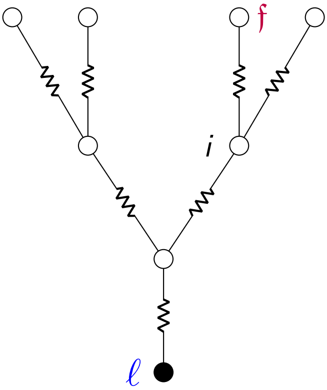
Electrically-inspired Message-Passing Algorithm

Electrical analogy (assuming $C^\top = C$)



Equilibrium opinions **x** are
the potentials of an electrical network

Electrical analogy (assuming $C^\top = C$)



Equilibrium opinions $\mathbf{x}$ are the potentials of an electrical network

- node $\mathfrak{f}$ has potential 0
- node ℓ has potential 1
- conductances —ww— of value $C_{ij} = C_{ji}$ substitute each edge

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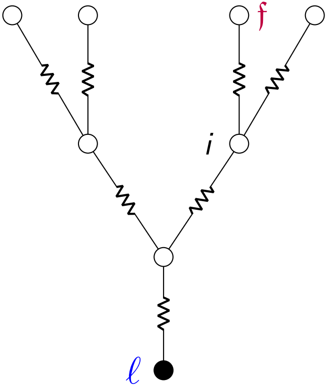

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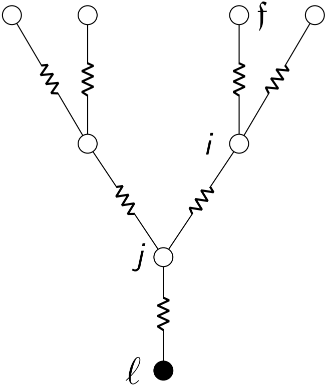
Computation of $H(\ell)$ on trees:

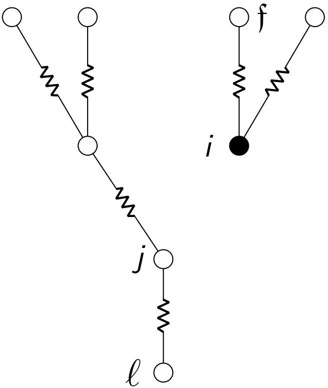
1. compute the effective resistances
2. compute the current leaving ℓ
3. compute all potentials
4. sum up potentials to get $H(\ell)$

Propagation of potentials: from leaves to root



Also $H(\ell)$ can be computed recursively, from the leaves to the root

Propagation of potentials: from leaves to root

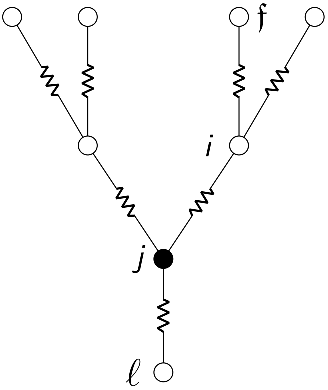


Also $H(\ell)$ can be computed recursively, from the leaves to the root

Notation:

- $H^{i\to j}$: $H(i)$ on the graph without edge $\{i,j\}$

Propagation of potentials: from leaves to root



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Notation:

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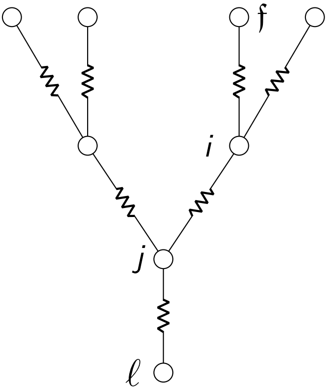
Propagation of potentials: from leaves to root



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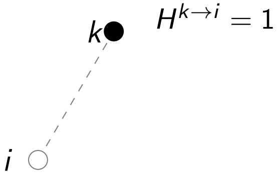
Propagation of potentials: example

For simplicity, $C_{ij} = 1$ for all $\{i, j\} \in E$



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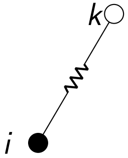

$H^{k \to i} = 1 \quad W^{k \to i} = 1$

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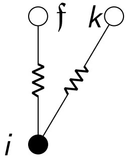

$$H^{k\to i} = 1 \qquad W^{k\to i} = 1$$
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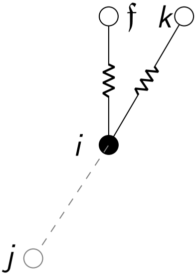

$$H^{k\to i} = 1 \quad W^{k\to i} = 1$$
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$$H^{i\to j} = 1 + W^{k\to i}H^{k\to i} + W^{\mathfrak{f}\to i}H^{\mathfrak{f}\to i}$$

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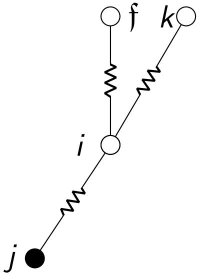

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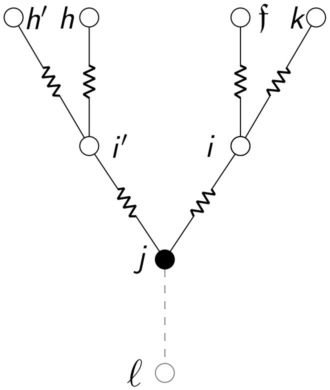

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$$H^{j\to\ell} = 1 + W^{i\to j} H^{i\to j} + W^{i'\to j} H^{i'\to j}$$
$$= 1 + W^{i\to j} + W^{i'\to j} + W^{k\to j} + W^{\mathfrak{f}\to j} + W^{h\to j} + W^{h'\to j}$$

because $W^{k\to j} = W^{k\to i} W^{i\to j}$

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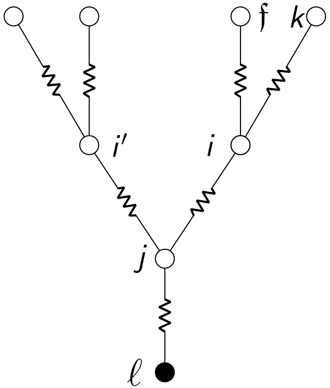

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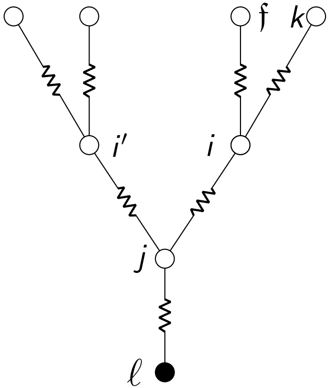

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$$W^{j\to \ell} = \frac{1}{1+(1-W^{i\to j})+(1-W^{i'\to j})}$$

$$H(\ell) = 1 + W^{j\to \ell} H^{j\to \ell}$$

Message Passing Algorithm

Generic graph $\mathcal{G} = (I, E, C)$ $\qquad$ C needs not be symmetric

Node i sends to neighbor j two messages:

- $W^{i\to j}(t)$: estimate of x_i if $\ell = j$
- $H^{i\to j}(t)$: estimate of $H(i)$ in the graph $\mathcal{G} \setminus \{i, j\}$

Message-Passing Algorithm

boundary $W^{\mathfrak{f}\to j}(t) = 0,\ H^{\mathfrak{f}\to j}(t) = 0$

initialization $W^{i\to j}(0) = 1,\ H^{i\to j}(0) = 1$

update
$$W^{i\to j}(t+1) = \left[1 + \sum_{k\in N_i^{-j}} \frac{C_{ik}}{C_{ij}} \left(1 - W^{k\to i}(t)\right)\right]^{-1}$$

$$H^{i\to j}(t+1) = 1 + \sum_{k\in N_i^{-j}} W^{k\to i}(t)\, H^{k\to i}(t)$$

estimate $H_t(\ell) = 1 + \sum_{i\in N_\ell} W^{i\to \ell}(t)\, H^{i\to \ell}(t)$

Analysis of the MPA

Convergence

Theorem

Let $\mathcal{G} = (I, E, C)$ be any connected graph with symmetric C.
Then, the Message Passing Algorithm converges

Proof outline:

1. define an MPA-like dynamics on directed graphs $\mathcal{M}$
2. define suitable message digraph $\mathcal{M}_{\mathcal{G}}$, that describes the topology of the dependences between messages
3. prove the convergence of the MPA-like dynamics induced on $\mathcal{M}_{\mathcal{G}}$:
   - when acyclic (by construction)
   - when strongly connected (more difficult)
   - in general (combining the sub-proofs)

Proof 1/3: MPA-like dynamics

- Digraph $\mathcal{M} = (V, \Phi)$, its adjacency matrix $M \in \{0,1\}^{V \times V}$
- Vectors $\mathbf{r}$, $\mathbf{s} \in \mathbb{R}^V_{>0}$, such that $r_v = s_v^{-1}$, and

$$W = \operatorname{diag}(\mathbf{r}) M \operatorname{diag}(\mathbf{s})$$

- Two sequences of non-negative vectors $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$, such that $\boldsymbol{\alpha}(t)$ is non-decreasing in every component and $\boldsymbol{\beta}(t)$ is convergent.

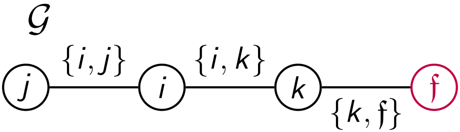
MPA-like is $\boldsymbol{\omega}(t) \in (0,1]^V$ and $\boldsymbol{\eta}(t) \in [1, +\infty)^V$ such that

$$\boldsymbol{\omega}(0) = \boldsymbol{\eta}(0) = \mathbf{1}$$

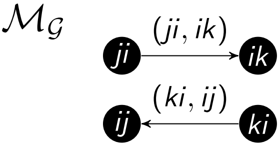
$$\omega_v(t+1) = \frac{1}{1 + \alpha_v(t) + \sum_w W_{vw} \left(1 - \omega_w(t)\right)}$$

$$\eta_v(t+1) = 1 + \beta_v(t) + \sum_w M_{vw}\, \omega_w(t)\, \eta_w(t)$$

Proof 2/3: Message digraph $\mathcal{M}_{\mathcal{G}}$



Social graph $\mathcal{G} = (I, E)$

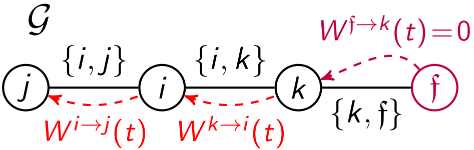


Message digraph $\mathcal{M}_{\mathcal{G}} = (\vec{E}, \Phi)$

$\vec{E} = \{ji : \{i,j\} \in E, i \neq \mathfrak{f}\}$

$\Phi = \{(ji, ik) : ji, ik \in \vec{E}, j \neq k\}$

Proof 2/3: Message digraph $\mathcal{M}_\mathcal{G}$



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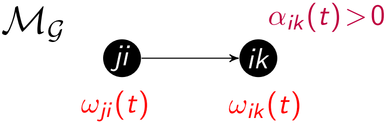


Message digraph $\mathcal{M}_\mathcal{G} = (\vec{E}, \Phi)$

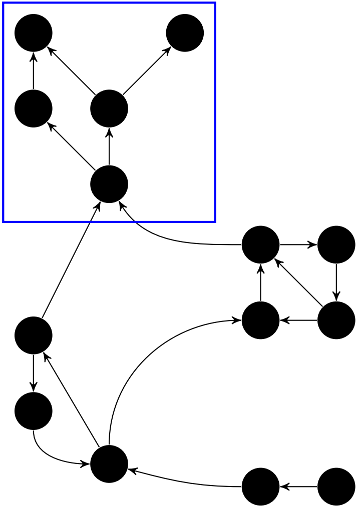
$\vec{E} = \{ji : \{i,j\} \in E, i \neq \mathfrak{f}\}$

$\Phi = \{(ji, ik) : ji, ik \in \vec{E}, j \neq k\}$

The messages $W^{i\to j}(t)$ and $H^{i\to j}(t)$ are associated to node ji in $\mathcal{M}_\mathcal{G}$

The counterpart of the constant message $W^{\mathfrak{f}\to k}(t) = 0$ is the (constant) sequence $\alpha_{ik} = C_{k\mathfrak{f}}/C_{ki} > 0$

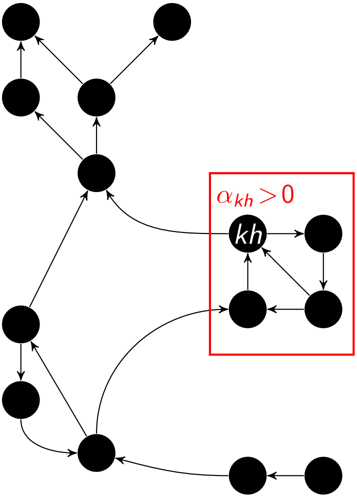
Proof 3/3: analysis on any digraph $\mathcal{M}$



If $\mathcal{M}$ acyclic $\implies$ convergence

(follow partial order)

Proof 3/3: analysis on any digraph $\mathcal{M}$

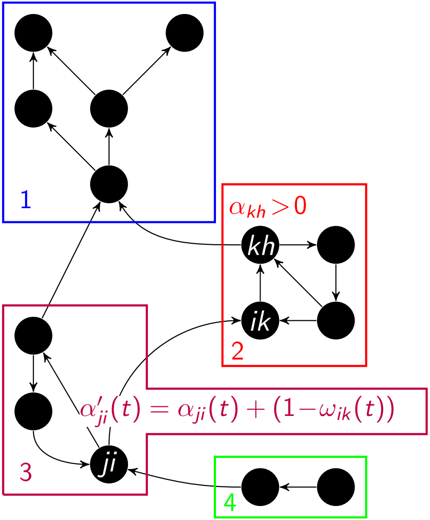


If $\mathcal{M}$ acyclic $\Longrightarrow$ convergence

If $\mathcal{M}$ is strongly connected and contains kh where $\alpha_{kh}(t) > 0$
$\Longrightarrow$ convergence

(W-messages have limits by monotonicity; update matrix for H-messages non-negative irreducible and eventually Shur stable)

Proof 3/3: analysis on any digraph $\mathcal{M}$

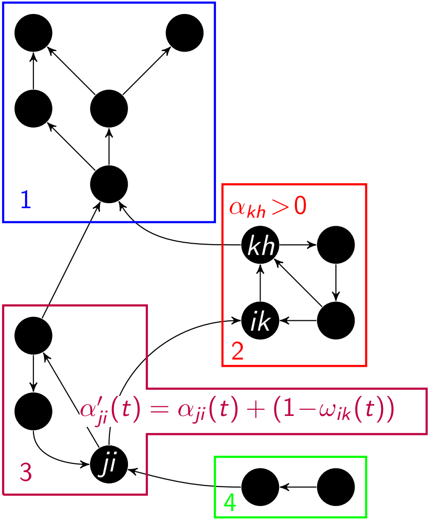


If $\mathcal{M}$ acyclic $\implies$ convergence

If $\mathcal{M}$ is strongly connected and contains kh where $\alpha_{kh}(t) > 0$ $\implies$ convergence

If every node in a non-trivial strongly connected component of $\mathcal{M}$ can reach kh where $\alpha_{kh}(t) > 0$ $\implies$ convergence

(condense components, use partial order, compose previous results)



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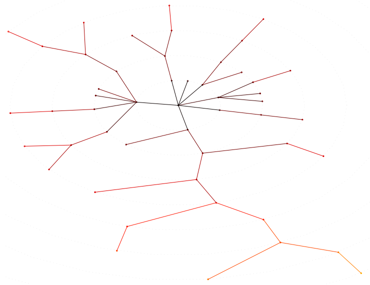
If every node in a non-trivial strongly connected component of $\mathcal{M}$ can reach kh where $\alpha_{kh}(t) > 0$ $\Longrightarrow$ convergence

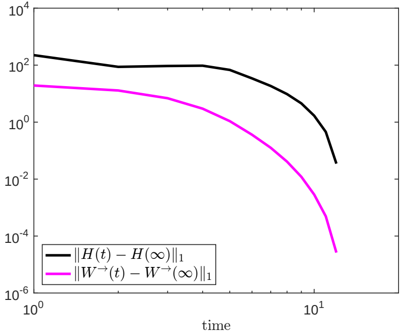
$\mathcal{M}_{\mathcal{G}}$ satisfies these assumptions $\Longrightarrow$ the MPA converges

Simulations

Simulations: random tree

Random tree graph: 50 nodes, 49 edges, diameter=13, $C_{i\mathfrak{f}} = 0.05$ for all i





Convergence time = diameter

Simulations: random tree

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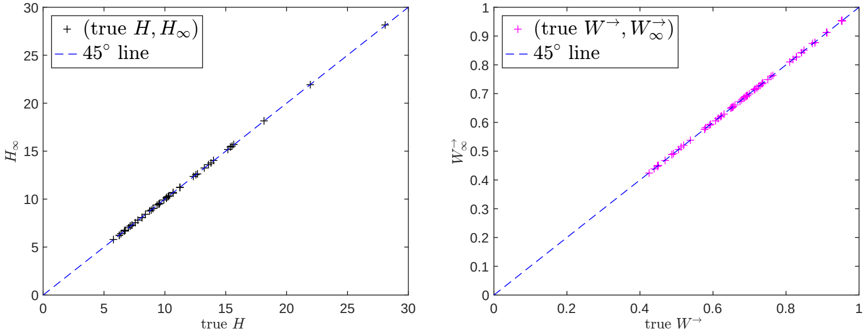
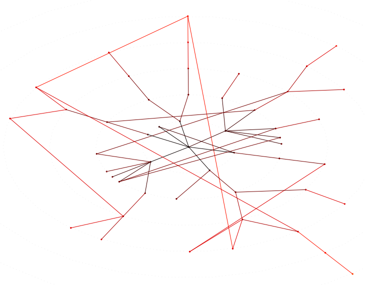


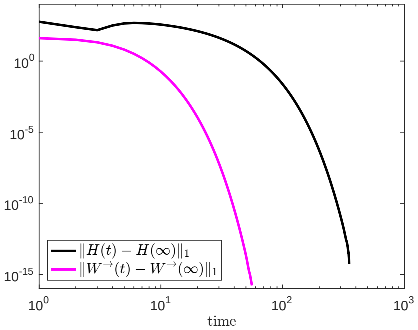
Left: true $H(\ell)$ vs. estimate $H_\infty(\ell)$ *Right:* true potential $W^{i\to\ell}$ vs. $W^{i\to\ell}_\infty$

MPA is *exact* on trees

Simulations: graph with few cycles

Random addition of 10 edges: 50 nodes, 59 edges, $C_{i\mathfrak{j}} = 0.05$

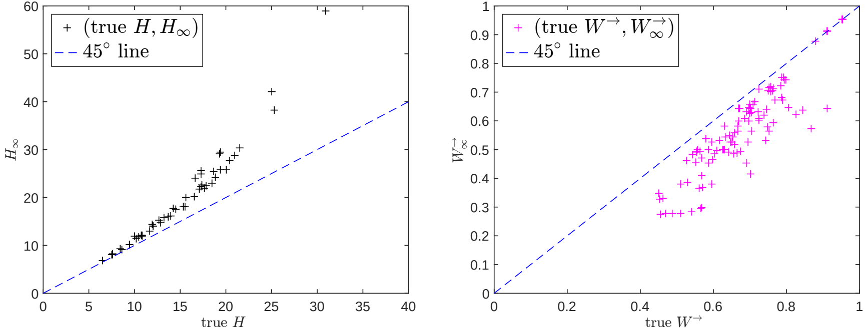




Convergence time of $W^{i\to j}(t)$ increases slightly
Convergence time of $H^{i\to j}(t)$ increases significantly

Simulations: graph with few cycles

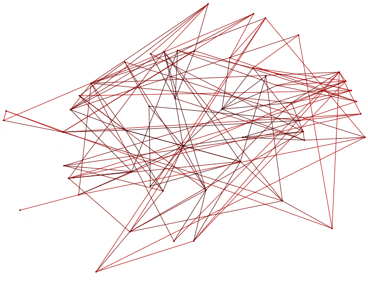
Random addition of 10 cycles: 50 nodes, 59 edges, $C_{i\mathfrak{f}} = 0.05$

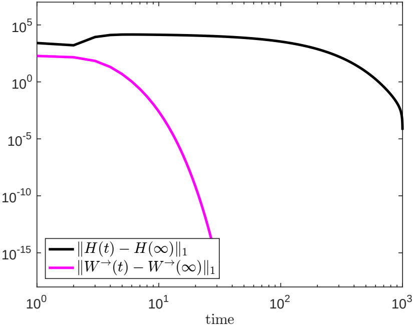


Left: true $H(\ell)$ vs. estimate $H_\infty(\ell)$ *Right:* true potential $W^{i\to\ell}$ vs. $W_\infty^{i\to\ell}$

Simulations: Erdős-Rényi random graph

Erdős-Rényi random graph: 50 nodes, 131 edges, $C_{i\mathfrak{f}} = 0.05$

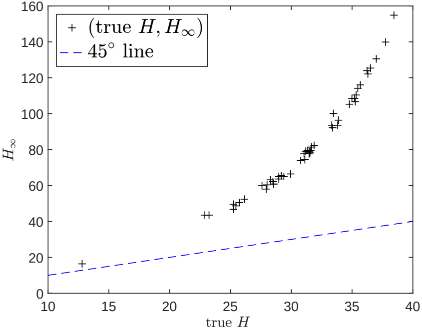


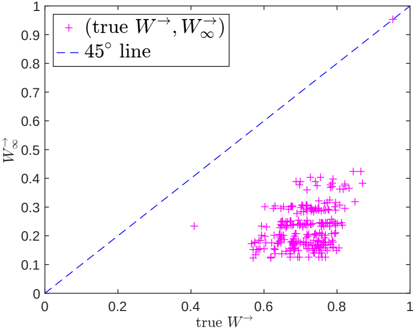


Convergence time of $W^{i\rightarrow j}(t)$ almost unchanges
Convergence time of $H^{i\rightarrow j}(t)$ increases significantly

Simulations: Erdős-Rényi random graph

Erdős-Rényi random graph: 50 nodes, 131 edges, $C_{i\mathfrak{f}} = 0.05$





Left: true $H(\ell)$ vs. estimate $H_\infty(\ell)$ *Right:* true potential $W^{i\to\ell}$ vs. $W_\infty^{i\to\ell}$

Conclusions

Summary: computing the Harmonic Influence

Message-passing algorithm with two messages H, W

- designed on trees by an electrical analogy
- can be used on any undirected weighted graph (I, E, C)
- proved to converge if $C^\top = C$
- convergence in two phases: first messages W, then H
  - cycles degrade convergence speed (of H)
- cycles degrade (not too much) the accuracy of the approximation

More insights in:

W.S. Rossi and P. Frasca. Mean-field analysis of the convergence time of message-passing computation of harmonic influence in social networks, IFACWC, Toulouse, 2017

Research outlook

Refine analysis of MPA

- Extend convergence proof to non-symmetric networks
- Evaluate convergence time
- Estimate the error between convergence value and actual H

Improve design of MPA

- Accelerate convergence of $H^{i \to j}$ messages

Can similar ideas be used for other centrality measures?