Stochastic processes with finite size scale invariance

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ABSTRACT

We present a theory of stochastic processes that are finite size scale invariant. Such processes are invariant under generalized dilations that operate on bounded ranges of scales and amplitudes. We recall here the theory of deterministic finite size scale invariance, and introduce an operator called Lamperti transform that makes equivalent generalized dilations and translations. This operator is then used to defined finite size scale invariant processes as image of stationary processes. The example of the Brownian motion is presented in some details to illustrate the definitions. We further extend the theory to the case of finite size scale invariant processes with stationary increments.

Keywords: scale invariance, finite size, stationary increments,

1. MOTIVATIONS AND AIMS

The property of scale invariance (also known as self-similarity) is a property shared by many natural or man-made nonlinear systems, as different as turbulent fluids, complex networks, Diffusion-Limited Aggregation clusters... Scale invariance is reminiscent of the concept of fractals, and is thus more and more recognized as a fundamental symmetry of Nature. However, this symmetry, like any, can be broken in different manners, and the breaking of scale invariance can lead to weaker forms of scale invariance, such as for example discrete scale invariance.[1,2] Another fundamental breaking of scale invariance is provided by the physical fact that scale invariance cannot exist for scale ranging from 0 to ∞: the invariance has to be broken by infra-red and ultra-violet cut-offs. Many examples of this breaking are known. One of the most famous is the Kolmogorov $k^{-5/3}$ law, which states that the power spectrum of the longitudinal increments of the velocity of a turbulent fluid scales as $k^{-5/3}$ in the limit of infinite Reynolds number.[3] However, the scaling is broken at finite Reynolds number,[4] and the scaling is restricted to the inertial range, roughly defined as the range of scales going from the integral scale (the larger scale of the experiment) to a small scale where energy is dissipated in heat.

These cut-offs are physically unavoidable, but generally difficult to be theoretically (and practically) taken into account. Usual approaches consider these limits in some sense like boundary conditions, therefore external to the physical laws. Another point of view, pioneered by L. Nottale,[5] is to incorporate the limits in the laws of physics, considering that scale, like time, is a physical quantity. Nottale's work, and later Dubrulle's and Graner's work[6,7] lead to the generalization of scale invariance to finite size scale invariance for which the cutoffs are part of the scaling laws. In the following, as performed by Dubrulle and Graner, we will consider physical fields that depend on physical variables, each quantity living in worlds limited by cut-offs. The difference with their work lies in the way we apply scale invariance: They work on the statistics of some random fields, whereas we directly work on the random fields themselves.

The paper is organized as follows. In the next section, we recall the necessary material on finite size scale invariance. This includes the definition of the laws, of the generalized dilation operators, and of deterministic scale invariance. In section 3, we show that finite size dilations are equivalent to translations thanks to an operator called Lamperti transform. In section 4, we define the class of finite size scale invariant processes and examine some of their properties. To illustrate the definition, we introduce the finite size scale invariant

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Brownian motion. In section 5, we go a little bit further by defining finite size scale invariant processes that have stationary increments: here, the finite size character is on the size of the increments. Finally, a discussion concludes the paper and gives some direction of our future work.

2. FINITE SIZE SCALE INVARIANCE

Let $X(t)$ be a field or a signal defined for $t \in \mathbb{T} =]T_-, T_+[$ with values in $\mathbb{X} =]X_-, X_+[$. We restrict here to one dimensional sets, we suppose that $\mathbb{T} \subset \mathbb{R}^+$, and therefore t is assumed positive; however, $\mathbb{X} \subset \mathbb{R}$ and the field under consideration can take positive and negative values. The bounds $T_\pm, X_\pm$ are the cut-offs that we want to take into account. Note that cut-offs are imposed both on the variable and the field.

Since we are going to generalize scale invariance, we obviously use multiplicative laws. Indeed, usual scale invariance or self-similarity with index H is defined as $X(\lambda t) = \lambda^H X(t), \forall \lambda$. We remark that the definition involves multiplicative laws, both for the variable t and the field X. To generalize the definition to finite size invariance, it is easier to use an additive representation of this definition. Thus we consider variable $\log t = a \in \mathbb{A} =]a_- = \log T_-; a_+ = \log T_+[$, variable which has the interpretation of a scale.[8] For the field, we will distinguish signed fields from unsigned fields. In the latter case, we restrict our attention to positive fields, and introduce $U(a) = \log X(\exp a) : \mathbb{A} \longrightarrow \mathbb{U} =]U_- = \log X_-, U_+ = \log X_+[$. In the additive representation, the usual scale invariance then writes $U(a + \log \lambda) - H \log(\lambda) = U(a)$. For signed fields, the bridge between multiplicative and additive representation is somewhat more difficult to build since the logarithm (in the field of real numbers) cannot be defined for negative values. Therefore, the introduction of a two-parameter representation is necessary. Using $\mathrm{Step}(x) = 0$ if $x \leq 0$ and $\mathrm{Step}(x) = 1$ otherwise, we define

$$(U(a) = \log |X(\exp a)|; \theta(a) = \mathrm{Step}(X(\exp a))) : \mathbb{A} \to \left\{ \begin{array}{ll}]-\infty, U_+ = \log X_+[& \text{if } \theta(a) = 0 \\ ]-\infty, U_- = \log -X_-[& \text{if } \theta(a) = 1 \end{array} \right\} \triangleq \mathbb{U} \times \mathbb{Z}/2\mathbb{Z} \quad (1)$$

If we want to generalize dilations to incorporate the cut-offs into the laws, we must insure that the interval on which we work ($\mathbb{T}, \mathbb{X}$ in the multiplicative representation, $\mathbb{A}, \mathbb{U}$ in the additive representation) are closed under dilations. In other words, the new dilation laws have to give to these intervals a group structure: existence of a unique identity element, existence of the inverse of a number, associativity of the law. If the law is commutative, the group is said to be Abelian.

2.1. Generalized addition

The laws we consider here are generalisations of the transformation law for velocities in special relativity.[6,7] Let $a_1, a_2 \in \mathbb{A} \times \mathbb{A}$, the new addition $\odot$ is defined as

$$a_1 \odot a_2 \quad = \quad \frac{a_1 + a_2 - a_1 a_2 (1/a_- + 1/a_+)}{1 - a_1 a_2 / a_- a_+}$$

This law gives to $\mathbb{A}$ a group structure: $\odot$ is associative, the identity element is 0, the inverse of a is $-a/(1 - a(1/a_- + 1/a_+))$. Furthermore, since the identity element 0 is in $\mathbb{A}$, then necessarily $a_- < 0$ and $a_+ > 0$. Let $S_\odot$ be the morphism associated to $\odot$, *i.e.* the bijection $S_\odot$:$(]a_-, a_+[, \odot) \longrightarrow (\mathbb{R}, +)$ such that $a_1 \odot a_2 = S_\odot^{-1}(S_\odot(a_1) + S_\odot(a_2))$. To obtain the explicit form of the morphism, we proceed as follows. We explicitly write the morphism equation,

$$S_\odot \left(\frac{a_1 + a_2 - a_1 a_2 (1/a_- + 1/a_+)}{1 - a_1 a_2 / a_- a_+} \right) = S_\odot(a_1) + S_\odot(a_2)$$

differentiate this expression with respect to a_2, and then set $a_2 = 0$. This leads to the following differential equation

$$S'_\odot(0) = S'_\odot(a) \Big(1 - a(1/a_- + 1/a_+) + a^2 / a_- a_+ \Big)$$

where the prime stands for the derivative. The solutions are easily shown to be

$$\begin{aligned} S_{\odot}(a) &= \frac{a_- a_+}{a_- - a_+} \log\left(\frac{1 - a/a_-}{1 - a/a_+}\right) \\ &= -a_{\pm} \log\left(1 - a/a_{\pm}\right) \text{ if } a_{\mp} \longrightarrow +\infty \\ &= a \text{ if furthermore } a_{\pm} \longrightarrow -\infty \end{aligned} \tag{2}$$

We hence recover the usual addition when the cut-offs go to infinity.

2.2. Law for positive fields

We can proceed for positive fields as for variables. To set notations, let $\otimes$ be the new addition of fields defined on the interval $\mathbb{U}$. The law reads

$$U_1 \otimes U_2 = \frac{U_1 + U_2 - U_1 U_2 (1/U_- + 1/U_+)}{1 - U_1 U_2 / U_- U_+}$$

and its associated morphism is $S_\otimes : (]U_-, U_+[, \otimes) \longrightarrow (\mathbb{R}, +)$. The morphism takes the same form as $S_\odot$ (see eq. 2), replacing a's by U's.

2.3. Law for signed fields

This case is a little bit more tricky, since as mentionned above, the additive representation of the field needs two parameters $(U, \theta) \in \mathbb{U} \times \mathbb{Z}/2\mathbb{Z}$. The new addition of fields $\otimes$ must give to $(\mathbb{U} \times \mathbb{Z}/2\mathbb{Z}, \otimes)$ a group structure. The generalization of the finite size addition to the case of this group is[9]

$$(U_1, \theta_1) \otimes (U_2, \theta_2) = \left(\frac{U_1 + U_2 - aU_1U_2 - b(\theta_1 U_2 + \theta_2 U_1) - c\theta_1\theta_2}{1 - dU_1U_2 - e\theta_1\theta_2}, \theta_1 + \theta_2 \right)$$

Of course, the definition (1) of the group constrains the previous law. Indeed, $(0, 0)$ is the identity element for $\otimes$; 0 in $\mathbb{R}$ is absorbing, and therefore its additive representation $(-\infty, 0)$ should also be absorbing : this implies $d = 0$; if $X_+ = X_-$, U and θ should be uncoupled: this implies $e = 0$; $(U_+, 0) \otimes (U_+, 0) = (U_+, 0)$, $(U_+, 0) \otimes (U_-, 1) = (U_-, 1)$ and $(U_-, 1) \otimes (U_-, 1) = (U_+, 0)$. All these constraints lead to some equations linking parameters $a, b, c, \ldots$; solving these equations leads to

$$(U_1, \theta_1) \otimes (U_2, \theta_2) = \left(U_1(1 - \theta_2 + \frac{U_-}{U_+}\theta_2) - \frac{U_1 U_2}{U_+} + U_2(1 - \theta_1 + \frac{U_-}{U_+}\theta_1) + \theta_1\theta_2(U_+ - \frac{U_-^2}{U_+}), \theta_1 + \theta_2 \right)$$

The morphism $S_\otimes : (\mathbb{U} \times \mathbb{Z}/2\mathbb{Z}, \otimes) \rightarrow (\mathbb{R}, +)$ is defined as S_+ and S_- depending on whether $\theta = 0$ or 1. Performing the same way as for $\odot$ allows to show that

$$S_\otimes [(U, \theta)] = \begin{cases} S_+(U) = -U_+ \log(1 - \frac{U}{U_+}) & \text{if } \theta = 0 \\ S_-(U) = -U_+ \log(\frac{U_- - U}{U_+}) & \text{if } \theta = 1 \end{cases}$$

2.4. Dilation operator

The usual dilation operator $\mathcal{D}^{mul}_{H, \log\mu}$ in the multiplicative representation is defined as

$$(\mathcal{D}^{mul}_{H,\lambda} X)(t) = \lambda^{-H} X(\lambda t)$$

which is written in the additive representation, if $\mu = \log\lambda$, as $(\mathcal{D}^{add}_{H,\mu} U)(a) = U(\mu + a) - H\mu$. In terms of composition laws, the dilation operator thus performs a dilation of the variable ($t \rightarrow \lambda t$ or $a \rightarrow a + \mu$) and a dilation (renormalization) of the field ($X \rightarrow \lambda^{-H} X$ or $U \rightarrow U - H\mu$). Likewise, the dilation operator with the bounded dilation laws $\odot, \otimes$ is defined as

$$(\mathcal{D}^{add}_{g,\mu} U)(a) = g(\mu) \otimes U(\mu \odot a)$$

where the renormalization function g has to be specified (see below). This operator is thus a finite size dilation operator. We are now ready to study the invariance of a field under the action of such an operator.

2.5. Scale invariance

A field is scale invariant* if it is equal to its dilated version. In terms of the additive representation, this reads

$$(\mathcal{D}^{add}_{g,\mu}U)(a) = U(a) = g(\mu) \otimes U(\mu \odot a) \tag{3}$$

This invariance principle has several consequences that we recall here. Applying two successive dilations by factors μ_1 and μ_2 to an invariant field allows to show that function g must satisfy

$$g(\mu_1 \odot \mu_2) = g(\mu_1) \otimes g(\mu_2)$$

Using the morphism $S_\otimes$, this equation implies

$$S_\otimes(g(\mu_1 \odot \mu_2)) = S_\otimes(g(\mu_1)) + S_\otimes(g(\mu_2))$$

meaning that function $S_\otimes \circ g$ is proportional to the morphism $S_\odot$ of $\odot$. For reasons that will become clear in a few lines, we set $(S_\otimes \circ g)(\mu) = -HS_\odot(\mu)$ and therefore, the normalization function is given by $g(\mu) = S_\otimes^{-1}(-HS_\odot(\mu))$.

Finally, the deterministic field that satisfies the scale invariance can be obtained by solving $U(a) = g(\mu) \otimes U(\mu \odot a)$ (see eq. 3). This is done by setting $a = 0$ to obtain

$$U(a) = U(0) \otimes g^{-1}(a) = \frac{-g(a)}{1 - g(a)(1/U_+ + 1/U_-)}$$

where g^{-1} is the inverse of g for $\otimes$ and where we have arbitrarily set $U(0) = 0$ on the r.h.s. of the last equation (this can be done by an appropriate choice of the origin in the additive representation, or the correct choice of units in the multiplicative representation). Figure 1 depicts these invariant fields for the nine generic cases that we can encounter, depending on whether a_-, a_+, U_-, U_+ are finite or not. In the case of no cut-off, we of course recover the usual power law or straight line in the additive representation $U(\mu) = H\mu$. This straight line is added in the eight other cases to illustrate the departure from usual scale invariance. Further, we can define a local similarity exponent as the derivative of the invariant field to obtain

$$\frac{dU}{da} = \frac{-g'(a)}{(1 - g(a)(1/U_+ + 1/U_-))^2}$$

which can be shown to be equal to H for $a = 0$. This self-similarity exponent could be interpreted wrongly in some experiment where the field is observed on a small range of scales. To illustrate that point, suppose that the finite interval $]a_-, a_+[$ contains the observation interval. Then the finite size scale invariant field is close to a straight line. This is illustrated for some cases in figure 2 where the range of scales on which the field is defined is either $\mathbb{R},]-6, +\infty[$ or $]-6, 6[$, all of these intervals containing the interval of scale $]-5, 5[$ where the field is measured. It is clear in these figures that the field is close to the usual self-similar field; the effect could be even more dramatic in case of experimental measurements.

This theory has been used with some success[7] to study avalanches in spin systems or cascades in turbulence. The theory is directly applied to deterministic functions, which in those applications are statistics of some random fields. The approach we adopt now is different in that we are going to work directly on the random fields, imposing them to be invariant under finite size dilation operator. But before doing that job, we need to introduce a transform which will put into correspondence the stochastic processes we will define and stationary processes.

3. GENERALIZED LAMPERTI TRANSFORMATION

A well-known result due to J. Lamperti and dating back to 1962[10] says that the stochastic process $X(t) = (\mathcal{L}_H Y)(t) = t^H Y(\log t)$ is scale invariant or self-similar of index H, *i.e.* $(\mathcal{D}_{H,\lambda}X)(t) = \lambda^{-H}X(\lambda t) \stackrel{d}{=} X(t)$, if Y is

*In this section, the discussion is implicitly restricted to the case of positive signals. The dilation operator in the case of signed signals will be presented in section 3.2.

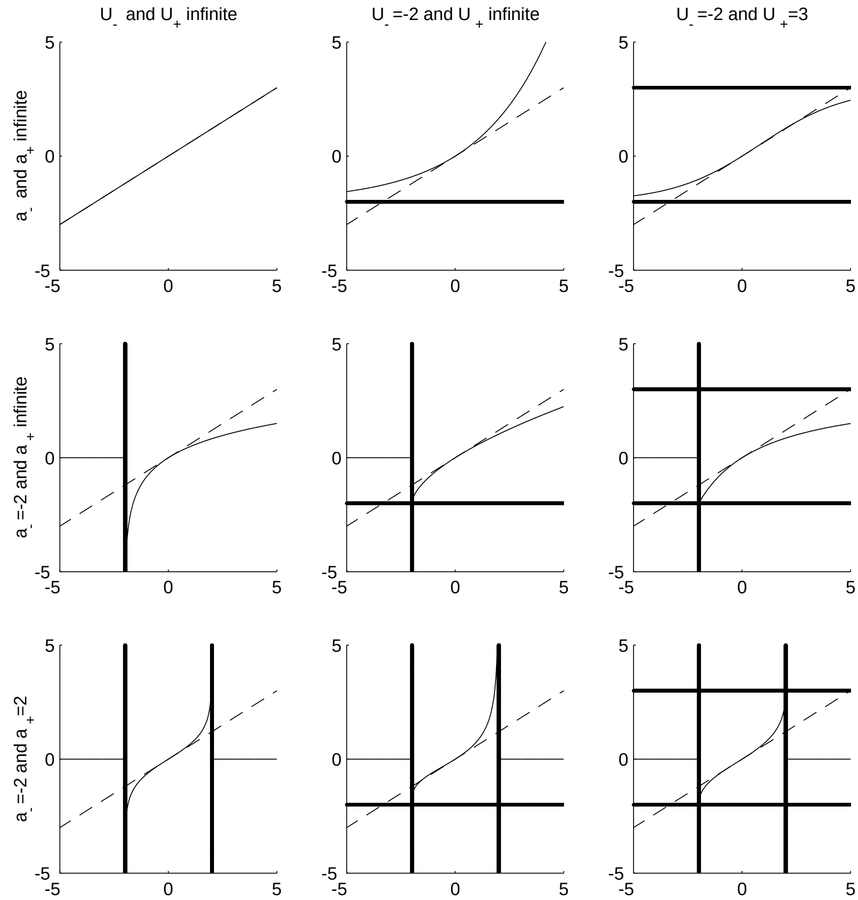

Figure 1. Scale invariant fields for the nine generic cases of finite or infinite cut-offs for the variable and the field. In the first row, the variable has no cut-off, in the second row, the variable is bounded from below and in the third row the variable has a low and a high cut-off. In the first column, the field is unbounded, whereas it is bounded from below in the second column and has two cut-offs in the third columns. The bold lines indicate the cut-offs.

stationary ($\stackrel{d}{=}$ stands for the equality in the sense of all finite dimensional distribution functions of the process.) The converse states that if X is self-similar with index H then $(\mathcal{L}_H^{-1}X)(t) = \exp(-Ht)X(\exp t)$ is stationary. Recalling that stationary means invariance under translation $\mathcal{T}_\tau$ of time lag τ, *i.e.* $(\mathcal{T}_\tau Y)(t) = Y(t+\tau) \stackrel{d}{=} Y(t), \forall\tau$, Lamperti's result states nothing but the equivalence of the translation operator and the dilation operator *via* the operator $\mathcal{L}_H$, or[11, 12]

$$\mathcal{L}_H^{-1}\mathcal{D}_{H,\lambda}\mathcal{L}_H = \mathcal{T}_{\log\lambda}$$

Using this equivalence offers a new reading of Lamperti's transform $\mathcal{L}_H$. It can be used to perform an operation on a self-similar process by working on its stationary equivalent process,[13] but it allows also to study some breaking of scale invariance by studying the equivalent breaking of stationarity. For example, stochastic discrete scale invariant processes have been defined as Lamperti transform of cyclostationary processes.[11, 12]

To define finite size scale invariant processes, we would like to keep the operator equivalence between translation and generalized dilation operators defined in the first section. Once again, we separate the case of positive fields and that of signed fields.

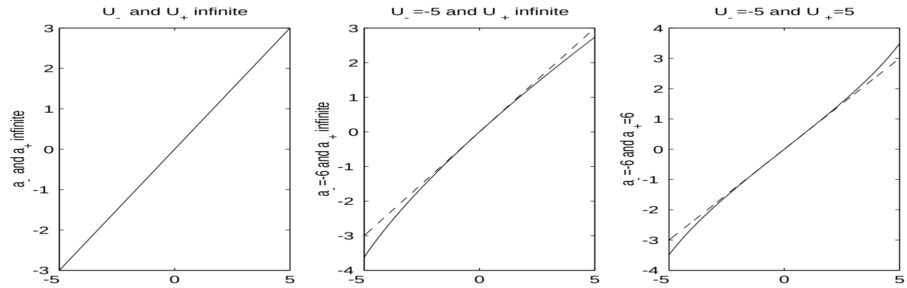

Figure 2. Illustration of the apparent usual self-similarity in the finite size scale invariant fields. For these pictures, the cut-offs on the variable are larger than the ends of the observation interval. The figures from the left to the right correspond to the diagonal of the previous figure.

3.1. Positive signals

Recall that the finite size scale dilation operator writes, in the additive representation,

$$(\mathcal{D}^{add}_{H,\mu}U)(a) = S_{\otimes}^{-1}(-HS_{\odot}(\mu)) \otimes U(\mu \odot a)$$

The operators

$$\begin{aligned}(\mathcal{L}^{add}_{H}Y)(a) &= S_{\otimes}^{-1}\left(Y(S_{\odot}(a))\right) \otimes S_{\otimes}^{-1}\left(HS_{\odot}(a)\right)\\ (\mathcal{L}^{-1,add}_{H}U)(t) &= S_{\otimes}\Big(U(S_{\odot}^{-1}(t))\Big) - Ht\end{aligned}$$

satisfy $\mathcal{D}^{add}_{H,\mu}\mathcal{L}^{add}_{H} = \mathcal{L}^{add}_{H}\mathcal{T}_{S_{\odot}(\mu)}$. This result can be verified by direct calculation and essentially makes use of the morphism definition $S_{\odot}(a \odot b) = S_{\odot}(a) + S_{\odot}(b)$.

By analogy with the usual situation, we call $\mathcal{L}^{add}_{H}$ the Lamperti transform associated to the dilation $\mathcal{D}^{add}_{H,\mu}$. The last thing to do is to write these operators in terms of the original variables X and t, it is to say to come back in the multiplicative representation. This is easy since $U(a) = \log X(\exp(a))$ and the results are

$$\begin{aligned}(\mathcal{D}^{mul}_{H,\mu}X)(t) &= \exp\left\{S_{\otimes}^{-1}(-HS_{\odot}(\mu)) \otimes \log X[\exp(\mu \odot \log(t))]\right\}\\ (\mathcal{L}^{mul}_{H}Y)(t) &= \exp\left\{S_{\otimes}^{-1}\left(\log Y(S_{\odot}(\log t))\right) \otimes S_{\otimes}^{-1}(HS_{\odot}(\log t))\right\}\\ &= \exp\left\{S_{\otimes}^{-1}\left\{\log Y(S_{\odot}(\log t)) + HS_{\odot}(\log t)\right\}\right\}\\ (\mathcal{L}^{-1,mul}_{H}X)(t) &= \exp\left\{S_{\otimes}\Big(\log X(e^{S_{\odot}^{-1}(t)})\Big) - Ht\right\}\end{aligned}$$

3.2. Signed signals

For signed signals, we have to discuss the definition of the dilation operator. Since the additive representation of signed signal is a two-parameter field (U, θ), the dilation operator reads

$$(\mathcal{D}^{add}_{g,\mu}U)(a) \quad = \quad (g(\mu), \gamma(\mu)) \otimes (U(\mu \odot a), \theta(\mu \odot a))$$

In the following, the renormalization function g is assumed to be of constant sign, and we choose arbitrarily $\gamma(\mu) = 0$, *i.e.*, the influence of g is the same whatever the sign of the field. This is in accordance with the usual renormalization λ^{-H}, and avoid strange behavior for which the sign of the renormalization fluctuates constantly. Function g can then be specified as in paragraph 2.5. We find $g(\mu) = S_{+}^{-1}(-HS_{\odot}(\mu))$, and the dilation operator explicitly writes

$$(\mathcal{D}^{add}_{H,\mu}U)(a) = \begin{cases} S_{+}^{-1}\left[-HS_{\odot}(\mu) + S_{+}\left(U(a \odot \mu)\right)\right] & \text{if } \theta(a) = 0\\ S_{-}^{-1}\left[S_{-}(S_{+}^{-1}(-HS_{\odot}(\mu))) + S_{-}\left(U(a \odot \mu)\right)\right] & \text{if } \theta(a) = 1\end{cases}$$

Lamperti transform and its inverse then write

$$(\mathcal{L}_H^{add} Y)(a) = \begin{cases} S_+^{-1}\Big(Y(S_\odot(a)) + HS_\odot(a)\Big) & \text{if } Y \geq 0 \\ S_-^{-1}\Big(-Y(S_\odot(a)) - S_-(S_+^{-1}(-HS_\odot(a)))\Big) & \text{if } Y < 0 \end{cases}$$

$$(\mathcal{L}_H^{-1,add} U)(t) = \begin{cases} S_+\Big(U(S_\odot^{-1}(t))\Big) - Ht & \text{if } \theta(a) = 0 \\ -S_-\Big(U(S_\odot^{-1}(t))\Big) - S_-(S_+^{-1}(-Ht)) & \text{if } \theta(a) = 1 \end{cases}$$

For the multiplicative representation, Lamperti transform and its inverse are then given by

$$(\mathcal{L}_H^{mul} Y)(t) = \begin{cases} \exp\Big\{S_+^{-1}\left[\log Y(S_\odot(\log t)) + HS_\odot(\log t)\right]\Big\} & \text{if } Y \geq 0 \\ -\exp\Big\{S_-^{-1}\left[\log -Y(S_\odot(\log t)) - S_-(S_+^{-1}(-HS_\odot(\log t)))\right]\Big\} & \text{if } Y < 0 \end{cases}$$

$$(\mathcal{L}_H^{-1,mul} X)(t) = \begin{cases} \exp\Big\{S_+\Big(\log X(e^{S_\odot^{-1}(t)})\Big) - Ht\Big\} & \text{if } X \geq 0 \\ -\exp\Big\{S_-\Big(\log -X(e^{S_\odot^{-1}(t)})\Big) + S_-(S_+^{-1}(-Ht))\Big\} & \text{if } X < 0 \end{cases}$$

Note that this expression simplifies if $U_- = U_+$ since then $S_- = S_+$, and Lamperti transform writes in that case $\mathrm{Sign}(Y(S_\odot(\log t)))\exp\Big\{S_+^{-1}\left[\log|Y(S_\odot(\log t))| + HS_\odot(\log t)\right]\Big\}$ (where $\mathrm{Sign}(x) = -1$ if $x < 0$, and $+1$ otherwise).

3.3. Lamperti in action

In the case of positive signals, the Lamperti transform can be written

$$(\mathcal{L}_H^{mul} Y)(t) = \exp\Big\{S_\otimes^{-1}\Big\{\log\Big[e^{HS_\odot(\log t)} Y(S_\odot(\log t))\Big]\Big\}\Big\}$$

and the effect of the transformation can be decomposed into two parts. The first step consists essentially in a time warping of the field Y. Precisely, this first step is the Lamperti transform associated with the case $U_\pm \to \infty$, since indeed in that case the Lamperti transform writes $(\mathcal{L}_H^{mul} Y)(t) = e^{HS_\odot(\log t)} Y(S_\odot(\log t))$. The second step consists then in bounding properly the time warped signal. In a way, the two actions are uncoupled. This will be of great use when studying stochastic processes. This interpretation remains in the general case of signed signals, but the writing is less elegant when $U_- \neq U_+$ because the symmetry between negative and positive parts is lost.

We are now ready to introduce stochastic processes with finite size scale invariance.

4. FINITE SIZE SCALE INVARIANT PROCESSES

Let $X(t)$ be a stochastic process indexed by $t \in \mathbb{T}$ with values in $\mathbb{X} \subset \mathbb{R}$. X is finite size scale invariant if

$$(\mathcal{D}_{H,\lambda}^{mul} X)(t) \stackrel{d}{=} X(t)$$

where once again $\stackrel{d}{=}$ stands for equality of all the finite dimensional distributions of the processes (*i.e.,* the distributions of multidimensional random variables $(X(t_1), \ldots, X(t_n))$). Before examining implications of the definition and developing examples, the previous paragraph allows to state:

Theorem: if the stochastic process $X(t)$ indexed by $t \in \mathbb{T}$ with values in $\mathbb{X} \subset \mathbb{R}$ is finite size scale invariant, then its inverse Lamperti transform $Y(t) = (\mathcal{L}_H^{-1,mul} X)(t)$ is a stationary process indexed by $\mathbb{R}$ with values in $\mathbb{R}$. The converse is true when Lamperti transforming the stationary process Y.

The proof of the theorem is easy and relies on the equivalence between finite size dilation operator and translation operator provided by the Lamperti transform. Furthermore, as for stationarity, we can define weaker forms of finite size scale invariance. For example, a process is second order finite size scale invariant if and only if it is the Lamperti transform of a second order stationary process. In the sequel, the inverse Lamperti transform of signal X will be called the stationary generator of X.

4.1. Some consequences

Let $X(t), t \in \mathbb{T}$ with values in $\mathbb{X}$ be a finite size scale invariant process with stationary generator $Y(t)$. We have seen that the Lamperti transform essentially begins by warping the time index (with a correct renormalization of the amplitude) and then warps the amplitude of the field.

The first step, time warping, then plays essentially on the correlation structure of the process. The second step, amplitude warping, acts on the distribution of probability of the process as a instantaneous nonlinearity.

One point probability description of the process. Suppose that the stationary process $Y(t)$ has a one point probability density function denoted as $P_Y(y)$, a function which is independent of t, otherwise Y would not be stationary. The Lamperti transform of Y can be written

$$\begin{aligned} X(t) &= \pm \exp\left\{ S_{\pm}^{-1}\big(\log g_{\pm}(\omega(t)) Y(\omega(t))\big) \right\} \text{ where} \\ g_-(x) &= -\exp\Big(- S_-\big(S_+^{-1}(-Hx)\big)\Big) \\ g_+(x) &= \exp Hx \end{aligned}$$

and where the subscript $\pm$ denotes the sign of the input $Y(\omega(t))$. We see here the warping $\omega(t) = S_\odot(\log(t))$ in time followed by the renormalization $g_\pm$. Let $Z_\pm(t) = g_\pm(\omega(t))Y(\omega(t))$. The one point probability density function of $Z_\pm$ is easily obtained and writes

$$P_{Z_\pm(t)}(z) = \frac{1}{\big|g_\pm(\omega(t))\big|} P_Y\Big(\frac{z}{g_\pm(\omega(t))}\Big)$$

Then we have $X(t) = \pm \exp S_\pm^{-1}\big(\log Z_\pm(t)\big)$ and the one point point density function of X follows

$$\begin{aligned} P_X(x) &= P_{Z_\pm(t)}\Big(\exp S_\pm(\log|x|)\Big) \frac{\exp\big(S_\pm(\log|x|)\big)\big|S'_\pm(\log|x|)\big|}{|x|} \\ &= \frac{\exp\big(S_\pm(\log|x|)\big)\big|S'_\pm(\log|x|)\big|}{\big|x g_\pm(\omega(t))\big|} P_Y\Big(\frac{\exp S_\pm(\log|x|)}{g_\pm(\omega(t))}\Big) \end{aligned} \tag{4}$$

where once again the $+$ subscript has to be selected when $x \geq 0$ and $-$ has to be selected if $x < 0$. We find again the two distincts effects of the time warping and the modulation warping : the first one plays linearly on the ampitude, whereas the second acts as a nonlinear transformation.

The two, three,... points probability density functions could be found in the same manner, but evidently calculations are more and more tricky since the distinction between negative and positive parts of the variables induces an exponential growth in the number of cases.

Covariance function. The covariance function of $X(t)$ is difficult to obtain in the general case. However, the unbounded amplitude case is simple, since the effect of the Lamperti transformation is only a time warping and a renormalization. Hence, if $S_+(U) = S_-(U) = U$, the Lamperti transform reads

$$(\mathcal{L}_H^{mul} Y)(t) = \exp(H\omega(t))Y(\omega(t))$$

and we have

$$\begin{aligned} C_X(t,s) &= \text{Cov}[X(t), X(s)] \\ &= \exp(H(\omega(t)+\omega(s)))\text{Cov}[Y(\omega(t)), Y(\omega(s))] \\ &= \exp(HS_\odot(\log t \odot \log s))\Gamma_Y\big(S_\odot(\log t \odot^{-1} \log s)\big) \end{aligned}$$

where $\Gamma_Y(\tau)$ is the covariance function of process Y, and as such, a non-negative definite function. Note that this expression generalizes easily to the case of multiple point correlation (so-called multicorrelations).

4.2. Finite size scale invariant Brownian motion

The well-known Brownian motion is a Gaussian process, with zero mean and variance equal to $\sigma^2 t$, and covariance $\mathrm{Cov}[B(t_1), B(t_2)] = \sigma^2 \min(t_1, t_2)$ for positive t_1, t_2. It is also known that Brownian motion is self-similar with index 1/2, *i.e.*, $B(\lambda t) = \lambda^{1/2} B(t)$. Since it is scale invariant in the usual sense, it admits a stationary generator *via* $B(t) = (\mathcal{L}_{1/2} Y)(t) = t^{1/2} Y(\log(t))$. Process Y is nothing but the Ornstein-Uhlenbeck process with covariance $\mathrm{Cov}[Y(t), Y(t+\tau)] = \sigma^2 \exp -|\tau|/2$. In this section, we are going to use the Ornstein-Uhlenbeck process to define a finite size scale invariant Brownian motion. To do so, we simply use the generalized Lamperti transform, and write

$$B_{fs}(t) = (\mathcal{L}^{mul}_{1/2} Y)(t)$$

for $t \in \mathbb{T} =]e^{a_-}, e^{a_+}[\in \mathbb{R}^+$, and B takes its values in $\mathbb{X} =] - e^{U_-}, e^{U_+}[$. By construction, we recover the usual Brownian motion as the limit of $B_{fs}(t)$ when the cut-offs go to infinity. We now present some features of the finite size scale invariant Brownian motion, by studying it for some values of the different cut-offs.

Brownian motion on the interval. Let $U_\pm$ go to infinity. We end up with a Brownian motion on the interval since the only finite size effect is the time warping. We can easily here study the correlation structure of the Brownian motion. As seen before, the correlation structure of the Lamperti transform is directly related to that of the stationary generator; in the case of interest in this example, we introduce the correlation function $R_X(t,s) = C_X(t,s)/\sqrt{C_X(t,t)C_X(s,s)}$, and we end up with

$$R_{B_{fs}}(t,s) = \exp\left\{ -\frac{|a_- a_+|}{2|a_- - a_+|} \left| \log \frac{1 - \log_{T_-}(t)}{1 - \log_{T_+}(t)} \times \frac{1 - \log_{T_+}(s)}{1 - \log_{T_-}(s)} \right| \right\} \tag{5}$$

where recall that $T_\pm = \exp a_\pm$, and where $\log_q$ stands for the logarithm in base q. The correlation functions for the three cases (no cut-off, one and two cut-offs) are plotted in figure 3. We can observe that the cut-offs do not modify that much the correlation structure of the Brownian motion (same decay), except for the case of a bounded interval for which the decay to zero is indeed observed. In the right of figure 3, we plot from top to bottom snapshots of the Brownian motion, Brownian motion with one cut-off and Brownian motion with two cut-offs. To obtain these plots, we use another interesting relation in the case $U_\pm \to \infty$, relation that links the usual scale invariant world with the finite size scale invariant world. Indeed, let $Y(t)$ be a stationary signal defined on $\mathbb{R}$ with values in $\mathbb{R}$. It can then be written $Y(t) = e^{-Ht} Z(e^t)$, where signal Z is a self-similar signal with parameter H. If we now make use of the generalized Lamperti transform, we define a finite size scale invariant process $X(t)$ with no cut-off in amplitude by $X(t) = \exp(H\omega(t)) Y(\omega(t)) = Z(\exp \omega(t))$. This allows to link easily usual self-similar signals to self-similar signals on the interval (in the sense of finite size scale invariance). The plots of figure 3 were obtained by performing the warping numerically on very long Brownian motions. Furthermore, this last expression allows another reading of the correlation function of the Brownian motion. Indeed, since $X(t) = Z(\exp \omega(t))$, we have $R_X(t,s) = R_Z(\exp \omega(t), \exp \omega(s)) = \exp(-\omega(t)/2 - \omega(s)/2) \min(\exp \omega(t), \exp \omega(s))$, an expression simpler than (5). .Note that the Brownian motion with two cut-offs explodes when approaching T_+: this effect corresponds to the fact that the usual Brownian motion goes to infinity when time goes to infinity. This explosion at a finite time could model critical phenomena where some fields exhibit such an explosion.

Bounded Brownian motion. We suppose now that there is no time warping and that the only finite size effect is due to the amplitude warping. This allows us to study the probabilistic nature of the motion. To illustrate this, we neglect the effect of the warping of time since as can be seen in equation (4), the warping appears through function $g_\pm$ which acts only as a multiplicative constant (with respect to the variable X), but depends on time. Since the Brownian motion is the Lamperti transform of the Ornstein-Uhlenbeck process, its one point probability density function can be obtain *via* equation (4) where $P_Y(.)$ is a Gaussian probability density. We draw two cases in figure 4. In the first case, we choose $-X_- \neq X_+$, implying that the nonlinear distorsion is asymmetric. The nonlinear function is depicted in the top left figure, whereas the Gaussian and the density of the transformed process are plotted in the top right figure.

In the case of a symmetric bounding $-X_- = X_+$ and no time warping, $S_\odot(x) = x$, the Lamperti transform is particularly simple,

$$(\mathcal{L}^{mul}_H Y)(t) \quad = \quad \mathrm{Sign}(Y(\log t)) \exp\left\{ S_+^{-1}\left[\log t^H |Y(\log t)|\right] \right\} = \mathrm{Sign}(Z(t)) \exp\left\{ S_+^{-1}\left[\log |Z(t)|\right] \right\}$$

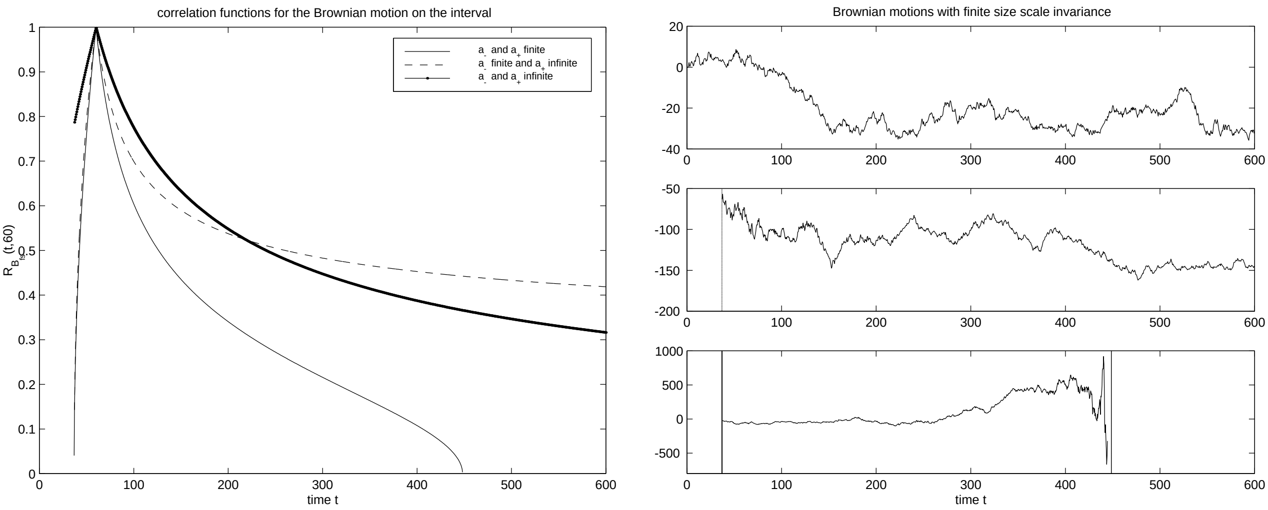

Figure 3. The figure on the left depicts a cut in the correlation functions of the Brownian motions with no cut-off (bold line), one cut-off in time (dashed line) and two cut-offs (continuous line). The figures on the right are snapshots of the Brownian motions (usual in the top, one cut-off in time in the middle, two cut-offs in the bottom).

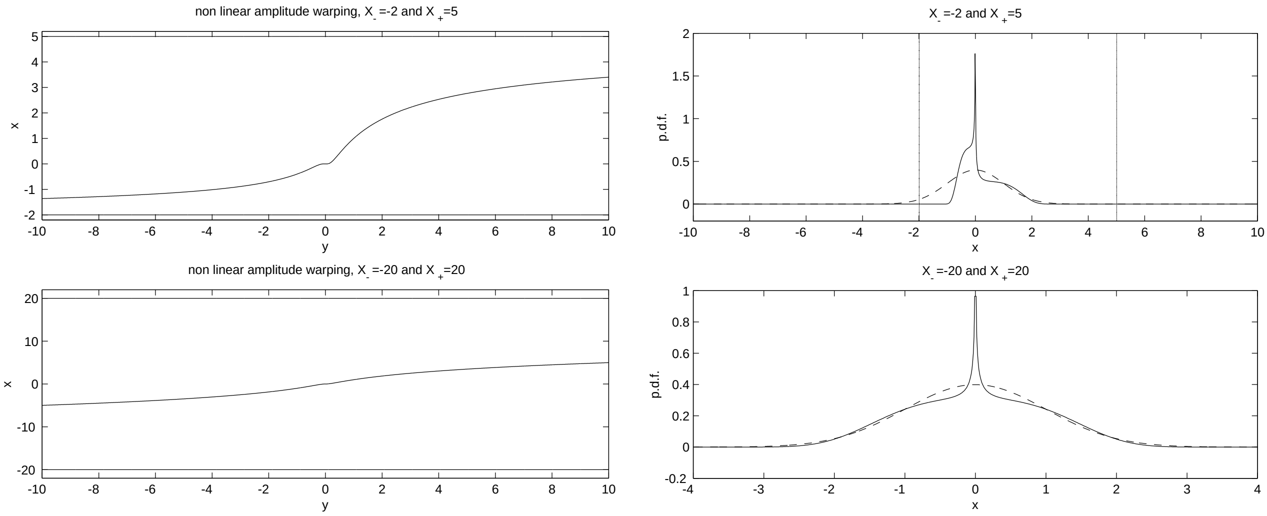

Figure 4. The figure on the left depicts the static nonlinearity used to warp the amplitude of the signals, for two cases. The figure on the right gives the corresponding probability density functions obtained, when the initial signal is Gaussian.

where as before, $Z(t) = t^H Y(\log(t))$ is the usual Lamperti transform of $Y(t)$. If $Y(t)$ is the Ornstein-Uhlenbeck process, then as we see before, $Z(t)$ is the usual Brownian motion, and we can easily simulate the bounded Brownian motion as a nonlinear transform of the usual Brownian motion. This is illustrated in figure 5.

5. FINITE SIZE SCALE INVARIANT PROCESSES WITH STATIONARY INCREMENTS

In the introduction, one of the physical motivations for defining stochastic processes with finite size scale invariance was turbulent velocity signals which seem to be scale invariant in a finite size range of scales. Scale in turbulent fluid measurements is usually defined as a length between two points in space (or in time when dealing with one measurement). Such a definition makes a scale defined as the inverse of a wavenumber (or a frequency).

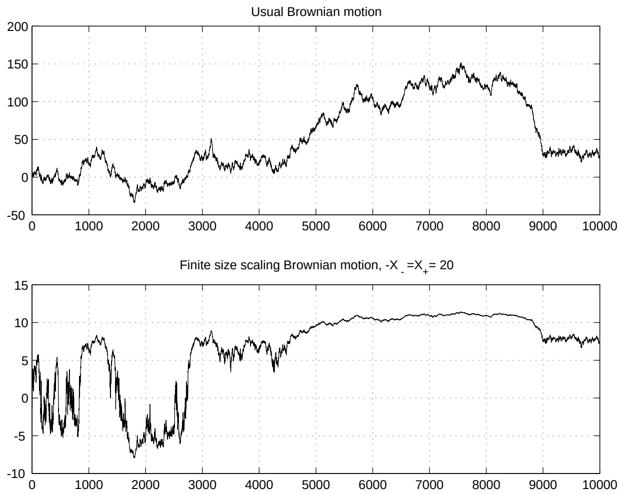

Figure 5. Snaphot of a Brownian motion and its finite size scale invariant counterpart.

The notion of scale we have worked with in the paper is a little bit different, since it relies on the ratio between two different times. This implies that the theory of finite size scale invariant deals with the study of processes defined on bounded range of times. The theory developed so far cannot handle the more usual notion of scale. Furthermore, to come back to the example of turbulence, it is rather well admitted and experimentally verified that the velocity field is non stationary but has stationary increments. It is therefore natural to go further in the development of our theory by considering finite size scale invariant processes with stationary increments.[14]

To define such processes, we need a two dimensional representation of a signal $Y(t) : \mathbb{R} \to \mathbb{R}$, such as the increments $Z(a,t) = Y(t+a) - Y(t)$, the wavelet transform,[15,16] $Z(a,t) = \int Y(u)\psi((u-t)/a)du/a$, or any other convenient forms. A well known notion is that of a self-similar process (of index H) with stationary increments, or Hss-si, defined for process Y as $\lambda^{-H} Z(\lambda a, \lambda t + \tau) \stackrel{d}{=} Z(a,t)$. A famous H-sssi process is the fractional Brownian motion popularized by Mandelbrot & Van Ness.[17] The definition puts in light a fundamental fact: the need of a two-parameter group of operators under the action of which the stochastic process is invariant. In the well known case of H-sssi, this group of operators is the group of time-scale displacements $(\mathcal{D}^H_{\lambda,\tau} Z)(a,t) = \lambda^{-H} Z(\lambda a, \lambda t + \tau)$, where the parameters $(a,t), (\lambda,\tau)$ live themselves in the group called the affine group. The affine group is the set $(\mathbb{R}^{+*} \times \mathbb{R})$ equipped with the law $\times$ defined by $(a_1,t_1) \times (a_2,t_2) = (a_1 a_2, t_1 + a_1 t_2)$. This law is not commutative and the affine group is not Abelian.

We now generalize the concept of H-sssi to that of finite size scale invariant process with stationary increments. To do so, we have to modify the affine group to take into account the finite size effect; for a general discussion on generalized affine groups, see the work of Hlawatsch.[18] Let the law $\oplus$ be defined as

$$(a_1,t_1) \oplus (a_2,t_2) = \big(\exp(\log a_1 \odot \log a_2), t_1 + \exp(S_\odot(a_1)) t_2\big) \tag{6}$$

where $\odot$ is the generalized addition of the previous sections. It can then be easily verified that $(\mathbb{X} \times \mathbb{R}, \otimes)$ is a group. The displacement operator in the time-scale plane (bounded in scale) is then defined as

$$(\mathcal{D}^H_{\lambda,\tau} Z)(a,t) = g(\lambda,\tau) \otimes Z\big((\lambda,\tau) \oplus (a,t)\big) \tag{7}$$

where $\otimes$ is the finite size addition of fields of the previous section. Note that we are implicitly in the additive representation for the amplitude. We can show, in the same manner as in the previous sections, that g does not depend on τ and writes $g(\lambda,\tau) = S_\otimes(-H S_\odot(\log \lambda))$.

A process $Y(t)$ is finite size scale invariant with stationary increments if its increments (or its wavelet transform) satisfy the stochastic equality $(\mathcal{D}^H_{\lambda,\tau} Z)(a,t) \stackrel{d}{=} Z(a,t)$. The consequences of this definition will not be presented in this paper since they are still under development.

6. DISCUSSION

The theory of finite size scale invariant stochastic processes developed in this paper has now to be confronted to physical situations. As presented here two classes of situations can be studied, depending on the definition of scale.

When scale is understood as the ratio between two different times, the theory amounts to work with signals defined on bounded intervals with values in bounded intervals. The laws of dilations are then finite size laws of dilations. Stochastic processes invariant under such generalized dilations could be used to model physical situations where critical times exist, such as for example fracture for which the time at which a material under some stress breaks up is a natural cut-off. In such an example, the cut-off is one of the parameters of physical system and is naturally considered by the approach developed here. Furthermore, an interesting extension of our theory would be the study of dynamical systems subject to the finite size scale invariance symmetry.

If scale is understood in a more usual way as the inverse of a frequency or the difference between two times, we sketch a construction of stochastic processes that are not only finite size scale invariant, but also do possess stationary increments. In this case, the description relies on two parameter group of operators that jointly and respectively dilate and translate signals in scale and time. Developments for this class of signals are under study and concern the possibility of defining such signals as some Lamperti transform, and the implications of finite size scale invariance on the statistics of the signals. Furthermore, we have the same ideas concerning the development of dynamical systems in scale for these processes.

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