LAMPERTI TRANSFORMATION FOR FINITE SIZE SCALE INVARIANCE

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ABSTRACT

In physical situations, scale invariance holds only for a limited range of scales. In this paper, one particular symmetry breaking of this kind is considered for stochastic processes using generalized composition laws of scales on intervals. This leads to the notion of finite size scale invariant stochastic processes. We show in particular that these processes are in correspondence with usual stationary processes *via* a generalized Lamperti transform. We illustrate our approach by introducing a finite size scale invariant version of fractional Brownian motion.

1. MOTIVATIONS

The property of scale invariance (also known as self-similarity) is a property shared by many natural or man-made systems, as different as turbulent fluids, complex networks, fractal aggregates, …

A field is scale invariant if it is identical to any of its rescaled version, up to some renormalization of its amplitude, or $(\mathcal{D}_{H,\mu}X)(t) \stackrel{\triangle}{=} \mu^{-H}X(\mu t) \stackrel{d}{=} X(t)$. For stochastic processes, the equality $\stackrel{d}{=}$ has to be understood in the sense of equality of all finite dimensional distributions. In 1962 [1], John Lamperti proved that the class of scale invariant processes can be put in correspondence with the class of stationary processes *via* the transformation $X(u) = (\mathcal{L}_H Y)(u) = u^H Y(\log u), u \in \mathbb{R}^{*+}$. Here, Y is a stationary process indexed by $\mathbb{R}$ whereas X is a self-similar process (with parameter H) indexed by $\mathbb{R}^{*+}$. This transformation is now called Lamperti transformation, and possesses an inverse defined as $(\mathcal{L}_H^{-1}X)(t) = \exp(-Ht)X(\exp(t))$.

A reading of Lamperti transformation in terms of operator theory shows that $\mathcal{L}_H$ makes the dilation operator $\mathcal{D}_{H,\mu}$ and the translation operator $\mathcal{T}_\tau$ equivalent according to

$$\mathcal{L}_H^{-1}\mathcal{D}_{H,\mu}\mathcal{L}_H = \mathcal{T}_{\log\mu}$$

This equivalence is fundamental and can be used in two important ways. The first one is to overcome the difficulties of some processing of a self-similar signal (*e.g.*, forecasting) by doing the processing on the equivalent stationary process (see [2] for instance). In the second way, the equivalence is exploited to study the so-called broken scale invariance.

Indeed, in many physical systems, scale invariance is only partially fulfilled, and we usually say that the invariance is broken. The word "partially" means that strict scale invariance does not exist, but that weaker form of scaling can remain. For example, some systems are not scale invariant continously (*i.e.*, $\forall\mu \in \mathbb{R}$) but rather discretely (*i.e.*, $\forall\mu = \mu_0^n, n \in \mathbb{Z}$) [3, 4]. This last case corresponds to the class of discrete scale invariant processes, which is in correspondence by Lamperti transformation with the class of cyclostationary processes [5].

Of importance here are systems for which it is argued that the invariance can not exist for the whole range of all possible values of scale a, but that there indeed exist a lower and an upper limit a_-, a_+. We also speak of cutoffs. These cutoffs are physically unavoidable, but generally difficult to be theoretically (and practically) taken into account. In turbulence for example, they are placed in the structure of the scale invariant process through functions assumed universal. These approaches hence consider these limits in some sense like boundary conditions, therefore external to the physical laws. Another point of view, pionneered by L. Nottale [6], is to incorporate the limits in the laws of physics, considering that scale, like time, is a physical quantity. Furthermore, observing an object at some scale means that this scale is relative to another, and therefore Nottale introduced the fundamental concept of scale relativity, one consequence of which is the existence of limiting scales if a special relativity point of view is adopted (like c, the speed of light, is a limiting speed). His work, further developed by Dubrulle and Graner [7], leads to the generalization of scale invariance to finite size scale invariance [8], for which the cutoffs are part of the scaling laws.

In this paper, we consider this approach, and we show that finite size scale invariant processes can be put in correspondence with stationary processes *via* a generalized Lamperti transformation. In section 2, we introduce the formalism of finite size scale invariance, and we derive in section 3 the associated Lamperti transform. Section 4 is devoted to the study of an example.

2. FINITE SIZE SCALING LAWS

The construction of finite size scale invariant processes relies on the use of additive representation groups over which signals and variables are defined. We therefore need (see below) to first work with positive signals and positive variables.

2.1. Positive signals and variables

Let $X(t)$ be a positive signal, where t is also assumed positive Define $U(a) = \log X(\exp(a))$ where $a = \log t$. This transform amounts to work with additive representations. For example, self-similarity with parameter H is written in these variables as $U(a) = U(a+\mu) - H\mu$.

The main idea to incorporate finite cutoffs in scale invariance is to generalize the law of composition of scales. After the previous change of variables, scale a lives in the group $(\mathbb{R}, +)$. If cutoffs a_-, a_+ exist, we have to work in the interval $]a_-, a_+[$, and find a law of composition $\odot$ such that $(]a_-, a_+[, \odot)$ has a group structure. In [7], it is shown that such a law exists and writes

$$a_1 \odot a_2 = \frac{a_1 + a_2 - a_1 a_2 (1/a_- + 1/a_+)}{1 - a_1 a_2 / a_- a_+}$$

Note that letting the cutoffs going to infinity gives back the addition.

In the usual scale invariance, the invariance is up to a renormalization of the amplitude. This renormalization is dependent on the law of composition of the fields themselves. Hence, the finite character of the invariance should also be included in the fields themselves. Therefore, we introduce two cutoffs $U_\pm$ for the fields, the law of composition for the fields becoming

$$U_1 \otimes U_2 = \frac{U_1 + U_2 - U_1 U_2 (1/U_- + 1/U_+)}{1 - U_1 U_2 / U_- U_+}$$

We are now ready to introduce the finite size scale invariance property as

$$(\mathcal{D}_{g,\mu} U)(a) \stackrel{\triangle}{=} g(\mu) \otimes U(\mu \odot a) \stackrel{d}{=} U(a) \qquad (1)$$

where we have introduced the dilation operator $\mathcal{D}_{g,\mu}$, and where we recover the usual scale invariance when the cutoffs go to infinity. The usual scale invariance is one of the nine generic cases that can happen depending on only wether $a_\pm$ is finite or not and $U_\pm$ is finite or not. The nine cases are characterized by functions g and the morphisms associated to $\odot$ and $\otimes$.

Let $S_\odot$:$(]a_-, a_+[, \odot) \longrightarrow (\mathbb{R}, +)$ and $S_\otimes : (]U_-, U_+[, \otimes) \longrightarrow (\mathbb{R}, +)$ be the associated group morphisms The mathematical form of these function can be shown to be

$$S_{\odot,\otimes}(x) = \frac{x_- x_+}{x_- - x_+} \log\Big(\frac{1 - x/x_-}{1 - x/x_+}\Big)$$

where $x = U$ or a. If $x_+ \to +\infty$, then $S_{\odot,\otimes}(x) = x_- \log(1 - x/x_-)$, and if furthermore $x_- \to -\infty$ then $S_{\odot,\otimes}(x) = x$ as expected. To obtain the form of the morphism, we write $S_\otimes(x \otimes y) = S_\otimes(x) + S_\otimes(y)$, take the derivative w.r.t. y and set $y = 0$. This gives a differential equation for $S_\otimes$, the solution of which is given above.

Function g satisfies the functional equation

$$g(a_1 \odot a_2) = g(a_1) \otimes g(a_2) \qquad (2)$$

To obtain this equation, set $a = 0$ in (1) to lead to $U(\mu) = U(0) \otimes g^{[-1]}(\mu)$ ($^{[-1]}$ stands for the inverse according to $\otimes$), and insert this relation back in (1). Then, using the definition of a group morphism and (2), we deduce $S_\otimes \circ g \propto S_\odot$, or equivalently

$$g(\mu) = S_\otimes^{-1}\big(-HS_\odot(\mu)\big)$$

The proportionality factor $-H$ has been chosen in order to recover the usual scale invariance when the cutoffs go to infinity.

2.2. Signed signals

In this section, we extend the preceding result to signals that take values in $\mathbb{R}$. We could also extend the theory to signed variables (time), but we are satisfied to work with positive variables : we match the usual approach of scale invariance which restricts time to the positive real line; furthermore, if scale is understood as the wavelet transform scale [9], then it is also sufficient to consider positive values.

Let $X(t)$ be a signal indexed by $\mathbb{R}^{*+}$ that takes its values in $\mathbb{R}$. For usual scale invariance, we consider a multiplicative group. However, $(\mathbb{R}, \times)$ is not a group since 0 does not have an inverse. Therefore, we work in a two parameter group by considering group $\mathbb{U}$, the elements of which write $(U(a), \theta(a))$, with $U(a) = \log|X(\exp(a))|$ and $\theta = 0$ if $X \geq 0$ and $\theta = 1$ if $X < 0$. This amounts to identify $\mathbb{R}$ with $\mathbb{R}^+ \times \mathbb{Z}/2\mathbb{Z}$. Furthermore, if X is confined in $]-X_-, X_+[$ where $X_\pm > 0$, then U is confined in $]-\infty, U_- = \log X_-[$ if $\theta = 1$ and in $]-\infty, U_+ = \log X_+[$ if $\theta = 0$. The law of composition in this two parameter group such that U lies in a finite interval has been shown to be [10]

$$(U_1, \theta_1) \otimes (U_1, \theta_2) = \Big(\frac{U_1 + U_2 - aU_1U_2 - b(\theta_1 U_2 + \theta_2 U_1) - c\theta_1\theta_2}{1 - dU_1U_2 - e\theta_1\theta_2}, \theta_1 + \theta_2\Big)$$

Constraints are of course needed to determine the values of the parameters $a, b, \ldots$: $(0,0)$ is the identity element for $\otimes$; 0 in $\mathbb{R}$ is absorbing, and therefore its additive representation $(-\infty, 0)$ should also be absorbing : this implies $d = 0$; if $X_+ = X_-$, U and θ should be uncoupled: this implies $e = 0$; $(U_+, 0) \otimes (U_+, 0) = (U_+, 0)$, $(U_+, 0) \otimes (U_-, 1) = (U_-, 1)$ and $(U_-, 1) \otimes (U_-, 1) = (U_+, 0)$. All these constraints implies that

$$\begin{aligned}(U_1, \theta_1) \otimes (U_1, \theta_2) &= \Big(U_1(1 - \theta_2 + \frac{U_-}{U_+}\theta_2) - \frac{U_1U_2}{U_+} \\ &+ U_2(1 - \theta_1 + \frac{U_-}{U_+}\theta_1) + \theta_1\theta_2(U_+ - \frac{U_-^2}{U_+}), \theta_1 + \theta_2\Big)\end{aligned}$$

Let $S_\pm$ be the morphism $(\mathbb{U}, \otimes) \to (\mathbb{R}, +)$. Using the same approach as in the preceding paragraph, it reduces to

$$S_\pm\left[(U, \theta)\right] = \begin{cases} S_+(U) = U_+ \log(1 - \frac{U}{U_+}) & \text{if } \theta = 0 \\ S_-(U) = U_+ \log(\frac{U_- - U}{U_+}) & \text{if } \theta = 1 \end{cases}$$

The dilation operator then write

$$(\mathcal{D}_{g,\mu}U)(a) \quad = \quad (g(\mu), \gamma(\mu)) \otimes (U(\mu \odot a), \theta(\mu \odot a))$$

In the following, the renormalization function g is assumed to be of constant sign, and we choose arbitrarily $\gamma(\mu) = 0$. Function g can then be specified as in the preceding paragraph. We find $g(\mu) = S_+^{-1}(-HS_\odot(\mu))$, and the dilation operator explicitly writes

$$\begin{array}{cr} (\mathcal{D}_{H,\mu}U)(a) = & \\ S_+^{-1}\left[-HS_\odot(\mu) + S_+\left(U(a \odot \mu)\right)\right] & \text{if } \theta = 0 \\ S_-^{-1}\left[S_-(S_+^{-1}(-HS_\odot(\mu))) + S_-\left(U(a \odot \mu)\right)\right] & \text{if } \theta = 1 \end{array}$$

We can now present the associated Lamperti transformations.

3. GENERALIZED LAMPERTI TRANSFORMATION

The idea of the Lamperti transform is to put in correspondance stationary processes with self-similar processes. Another way of saying this in the language of operators is the equivalence $\mathcal{D}_{H,\mu}\mathcal{L}_H = \mathcal{L}_H\mathcal{T}_{\log\mu}$ where $(\mathcal{T}_\tau Y)(t) = Y(t+\tau)$ is the translation operator. This equality in terms of operators means that an invariance under the action of one operator is transported to the other by the Lamperti transform.

The idea is therefore to build a generalized Lamperti transform such that the translation of a signal is transported to the finite scale size dilation. We again separate positive and negative fields.

3.1. Positive signals

Recall that the finite scale size dilation operator writes

$$(\mathcal{D}_{H,\mu}U)(a) = S_\otimes^{-1}(-HS_\odot(\mu)) \otimes U(\mu \odot a) \tag{3}$$

It is then easy to prove that the operator

$$(\mathcal{L}_H^U)Y(a) = S_\otimes^{-1}\left(\log Y(S_\odot(a))\right) \otimes S_\otimes^{-1}\left(HS_\odot(a)\right) \tag{4}$$

satisfies

$$\mathcal{D}_{H,\mu}\mathcal{L}_H^U = \mathcal{L}_H^U\mathcal{T}_{S_\odot(\mu)}$$

Therefore, we call $\mathcal{L}_H^U$ the Lamperti transform associated to the dilation $\mathcal{D}_{H,\mu}$. The last thing to do is to write these operators in terms of the original variable X and t. This is easy since $U(a) = \log X(\exp(a))$ and the results are

$$\begin{aligned} (\mathcal{D}_{H,\mu}X)(t) &= e^{S_\otimes^{-1}(-HS_\odot(\mu)) \otimes \log X[\exp(\mu \odot \log(t))]} \\ (\mathcal{L}_H Y)(t) &= e^{S_\otimes^{-1}(\log Y(S_\odot(\log t))) \otimes S_\otimes^{-1}(HS_\odot(\log t))} \\ &= e^{S_\otimes^{-1}\{\log Y(S_\odot(\log t)) + HS_\odot(\log t))\}} \end{aligned}$$

3.2. Signed signals

If the signals takes any sign, we have already presented the dilation operator. The Lamperti transform reads

$$\begin{array}{lr} (\mathcal{L}_H Y)(t) = & \\ e^{S_+^{-1}[\log Y(S_\odot(\log t)) + HS_\odot(\log t)]} & \text{if } Y \geq 0 \\ -e^{S_-^{-1}\left[\log -Y(S_\odot(\log t)) - S_-(S_+^{-1}(-HS_\odot(\log t)))\right]} & \text{if } Y < 0 \end{array}$$

Note that this expression simplifies if $U_- = U_+$ since in that case $S_- = S_+$, and Lamperti transform writes in that case $\text{Sign}(Y(S_\odot(\log t)))e^{S_+^{-1}[\log|Y(S_\odot(\log t))| + HS_\odot(\log t)]}$.

4. STOCHASTIC PROCESSES WITH FINITE SIZE SCALE INVARIANCE

Let $X(t)$ be a stochastic process indexed by an interval $\mathbb{T} =]T_-, T_+[\subset \mathbb{R}^{*+}$, which takes its values in the interval $\mathbb{X} =]-X_-, X_+[\subset \mathbb{R}$. The process is said to be scale invariant if

$$(\mathcal{D}_{H,\mu}X)(t) \stackrel{d}{=} X(t) \tag{5}$$

If one of the parameters $X_\pm, T_\pm$ is finite then the process will be said finite size scale invariant. In that case, and according to the general construction, there exists a stationary process $Y(t)$ indexed by $\mathbb{R}$ with values in $\mathbb{R}$ such that $X(t) = (\mathcal{L}_H Y)(t)$. The effect of Lamperti transform on process Y is three fold. Firsly, time is warped by the warping $u \in \mathbb{R} \to t = \exp S_\odot^{-1}(u) \in \mathbb{T}$. Secondly, the process is bounded to live in $\mathbb{X}$. This bounding is a static nonlinear transformation, and this induces of course a change of probability measures. Finally, there is the renormalization by $g(\mu)$. These three steps makes the study of a finite size scale invariant process difficult in general. In the sequel, we concentrate on two examples.

4.1. Fractional Brownian motion

In this example we consider a generalization of fractional Brownian motion (fBm) [11, 12]. The fBm is the Gaussian process with stationary increments and H-self-similar. Its covariance $C_{B_H}(t,s) = \text{Cov}[B_H(t), B_H(s)]$ is $\sigma^2/2(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$. Since fBm is self-similar, it is the usual Lamperti transform of some stationary process Y_H whose covariance function is given by $R_{Y_H}(\tau) = \sigma^2(\cosh(H|\tau|) - 2^{2H-1}|\sinh(\tau/2)|^{2H})$. Let us now impose cutoffs to the scaling laws. We do not impose cutoffs on the magnitude on the fields ($U_\pm \to \infty$) so that the generalized Lamperti transform of Y_H reads

$$(\mathcal{L}_H Y_H)(t) \quad = \quad Y_H(S_\odot(\log t)) \exp HS_\odot(\log t) = Z_H(t)$$

The covariance function of Z_H reads

$$\frac{\sigma^2}{2}\{e^{2HS_\odot(\log t)} + e^{2HS_\odot(\log s)} + |e^{S_\odot(\log t)} - e^{S_\odot(\log s)}|^{2H}\}$$

for $(t,s) \in]T_-, T_+[^2$. Furthermore, the process remains a zero mean Gaussian process, and has a variance proportional to $\exp 2HS_\odot(\log t)$. Note that we recover the properties of the fBm when $T_- \to 0$ and $T_+ \to +\infty$. This construction is illustrated in figure (1) where we plot the fBM for $H = 0.7$, its stationary generator Y_H, and its finite size scale invariant counterpart Z_H. Note that the particular form of this process: its variance is going to infinity as time approaches its upper limit T_+; it could be a tentative model of some critical phenomena ending in a crash at T_+. Note that for $H = 1/2$, $Y_{1/2}$

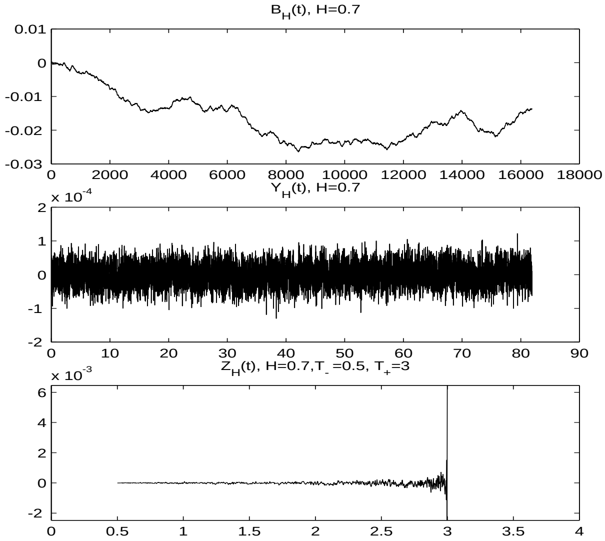

Fig. 1. Trace of fBm (top), its stationary counterpart in the middle (obtained by the usual inverse Lamperti transform), and the finite size scale invariant fBm.

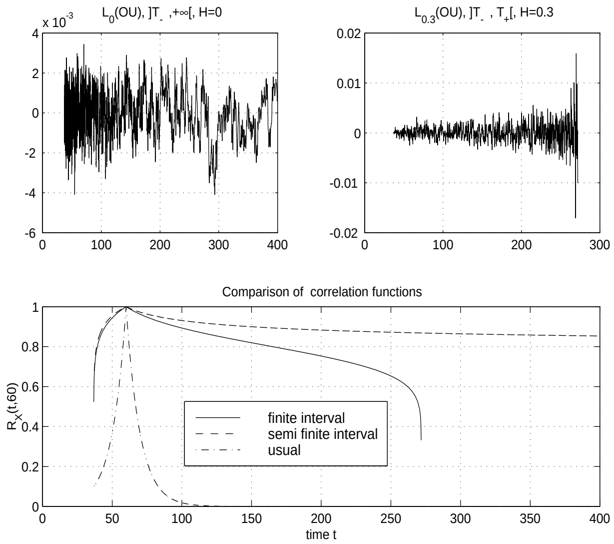

Fig. 2. Illustration of finite scale size invariance (see text).

reduces to the Ornstein-Uhlenbeck process, whose covariance function reads $\sigma^2 \exp(-|\tau|/2)$. Since two cutoffs T_-, T_+ are placed in time, it can be shown that

$$S_{\odot}(\log t) = \log \left\{ \frac{1 - \log_{T_-}(t)}{1 - \log_{T_+}(t)} \right\}$$

and the process $Z_{1/2}$ can be called a finite size scale invariant Brownian motion. Working with the correlation function $R_X(t,s) = C_X(t,s)/\sqrt{C_X(t,t)C_X(s,s)}$, we end up for the transformed Ornstein-Uhlenbeck process $Z_{1/2}$ with

$$R_X(t,s) = \exp\left\{ -\frac{1}{2} \left| \log \frac{1 - \log_{T_-}(t)}{1 - \log_{T_+}(t)} \times \frac{1 - \log_{T_+}(s)}{1 - \log_{T_-}(s)} \right| \right\}$$

To illustrate all this, we plot in figure (2) two examples of Lamperti transformed Ornstein-Uhlenbeck process :

1- top-left : $H = 0$, T_- is finite, T_+ is infinity.

2- top-right : $H = 0.3$, $T_\pm$ are finite.

These signals are invariant under their corresponding finite scale size dilation operator. The bottom panel displays the correlations of the Ornstein-Uhlenbeck process (usual), and the two cases finite or semi finite intervals. Note that we have translated the usual correlation function at time 60, since we plot $R_X(t, 60)$ for the two other processes. Note that the finite scale size invariant processes seem to be more long range dependent than the usual (mixing) Ornstein-Uhlenbeck process. This fact has to be studied further.

5. CONCLUSIONS

We have shown here how the Lamperti transformation can be extended to generalized scaling laws. Finite size scale invariant stochastic process may be important in physical applications where scaling relations holds only for a finite range of scales, or, as illustrated here, a finite range of time. We are currently investigating a generalization of the construction made here to finite size scale invariant processes with stationary (in a generalized sense) increments.

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