STOCHASTIC DISCRETE SCALE INVARIANCE AND LAMPERTI TRANSFORMATION

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ABSTRACT

We define and study stochastic discrete scale invariance (DSI), a property which requires invariance by dilation for certain prefered scaling factors only. We prove that the Lamperti transformation, known to map self-similar process to stationary processes, is an important tool to study these processes and gives a more general connection: in particular between DSI and cyclostationarity.
Some general properties of DSI processes are given. Examples of random sequences with DSI are then contructed and illustrated. We address finally the problem of analysis of DSI processes, first using the inverse Lamperti transformation to analyse DSI processes by means of cyclostationary methods. Second we propose to re-write these tools directly in a Mellin formalism.

1. DISCRETE SCALE INVARIANCE

Scale invariance, also called self-similarity, is frequently called upon. Its central point is that the signal is scale invariant if it is equivalent to any of its rescaled versions, up to some amplitude renormalization [1]. More precisely, a function $X(t)$ is scale-invariant with exponent H, or H-ss, if for any $k \in \mathbb{R}$: $X(kt) = k^H X(t)$.

This definition is given here for a deterministic signal. The concept is extended to stochastic signals when one thinks of the previous equality in a probabilistic way: the equality of the finite-dimensional probability distributions [1]. We will write $\stackrel{d}{=}$ this equality.

The strict notion of scale invariance, valid for all dilation factors above, is in some cases too rigid; the middle-third Cantor set is for example invariant only by dilations of a factor 3 (or a power of 3). Several weakened versions of self-similarity have been proposed to enlarge scale invariance's relevance and one is of special interest here : it is to require invariance by dilation for certain prefered scaling factors only, the same property than the Cantor set. This is known as *discrete scale invariance* (DSI), a concept which as been stressed upon by Sornette and Saleur [2, 3] as an efficient model in many situations (fracture, DLA, critical phenomena, earthquakes).

They studied DSI as a property of deterministic signals. and general arguments as why should DSI naturally occurs: classical scenarii involve the existence of a characteristic scale, the apparition by instability of a prefered scale or more general arguments in non-unitary field theories [4]. They also found ways to estimate the prefered scaling ratio in this context, based on classical spectral analysis (Lomb periodogram).

As far as we know, this property has not been envisaged for stochastic processes which is often fruitful to dispose of when dealing with real measurements, as it allows to use statistical signal processing methods. The extension of DSI property to stochastic processes is straigthforward. We propose the following definition:

A process $\{X(t),\, t \in \mathbb{R}^+\}$ has discrete scale invariance with scaling exponent H and scale λ if:

$$X(\lambda t) \stackrel{d}{=} \lambda^H X(t),\ t \in \mathbb{R}^+.$$

We will refer to this property as (H, λ)-DSI. The equality here is the probabilistic equality. In the following only wide-sense property will be used (second-order statistical properties only).

2. LAMPERTI TRANSFORM : DSI AS AN IMAGE OF CYCLOSTATIONARITY

2.1. Lamperti transformation

A main issue is to find a way to study both theoretically and practically DSI processes. The answer is given by a transformation introduced by J. Lamperti in 1962 [5], which is an isometry between self-similar and stationary processes. It will be called the Lamperti transformation and is defined as follows.

For any process $\{Y(t),\, t \in \mathbb{R}\}$, its Lamperti transform $\{X(t),\, t \in \mathbb{R}^+\}$ and its inverse are given by

$$X(t) = (\mathcal{L}Y)(t) \ \hat{=}\ t^H Y(\ln t),\ \ t \in \mathbb{R}^+; \quad (1)$$
$$Y(t) = \left(\mathcal{L}^{-1}X\right)(t) \ \hat{=}\ e^{-Ht} X(e^t),\ \ t \in \mathbb{R}. \quad (2)$$

The theorem in the paper of Lamperti is that a process $Y(t)$ is stationary if and only if its Lamperti transform $X =$

$\mathcal{L}Y$ is H-ss. The cebtral argument of the demonstration is that the Lamperti transformation maps a time-shifted process to the dilated version of the Lamperti transform of the original process. Let $\left(\mathcal{D}_\lambda^H X\right)(t) \hat{=} \lambda^{-H} X(\lambda t)$ be the dilation operator and
$\left(\mathcal{S}_\tau Y\right)(t) \hat{=} Y(t+\tau)$ the time-shift operator. The property is that

$$\left(\mathcal{L}^{-1}\ \mathcal{D}_\lambda^H\ \mathcal{L}\, Y\right)(t) \stackrel{d}{=} \left(\mathcal{S}_{\ln\lambda} Y\right)(t). \qquad (3)$$

Understanding this correspondence between time-shift and dilation operators, we can propose many variations around Lamperti's theorem, relaxing in some way the stationarity for Y and the self-similarity for X. We will only consider here the DSI property but some results about different classes of processes and their description are proposed in [6]. A usefull property is that one can give the (potentially nonstationary) correlation function of the Lamperti transform X of a process Y:

$$\mathbb{E}\left\{X(t)X(s)\right\} \hat{=} R_X(t,s) \;=\; (st)^H R_Y(\ln t, \ln s). \quad (4)$$

In the recent years some results have been obtained for H-ss processes with this transformation. Yazici and Kashyap proposed a general description of wide-sense self-similar processes and linear models for H-ss [7]. Burnecki *et al.* study α-stable and H-ss processes with this transform [8]. Nuzman and Poor give important results about the prediction, the whitening and the interpolation of H-ss processes, mainly applied to the fractional Brownian motion [9]. Finally Vidács and Virtamo [10] proposed a method of estimation of H for a fBm based on the same idea. All these authors use the inverse Lamperti transformation (2) to map the question to a stationary problem and then use the known results for stationary issues in this context. Our objective is to show that nonstationary methods can be adapted in the same way, especially for DSI.

2.2. DSI and cyclostationarity

A process is called cyclostationary [11] or periodically-correlated [12, 13], if its correlation function is periodic in time. More precisely, if a period T is given, a process $\{Y(t),\ t \in \mathbb{R}\}$ is wide-sense cyclostationary if it satisfies for any times t, s

$$\begin{aligned} \mathbb{E}\,Y(t+T) &= \mathbb{E}\,Y(t), \\ \mathbb{E}\{Y(t+T)Y(s+T)\} &= \mathbb{E}\{Y(t)Y(s)\}, \end{aligned} \quad (5)$$

The correlation function $R_Y(t, t+\tau)$ is then periodic in t of period T and one can decompose R_Y in a Fourier series:

$$R_Y(t, t+\tau) \;=\; \sum_{n=-\infty}^{+\infty} C_n(\tau) e^{i2\pi n t/T}. \qquad (6)$$

Using the definitions of cyclostationarity and (H,λ)-DSI and the correspondance (3), we can state the following important result.

A process $\{Y(t),\ t \in \mathbb{R}\}$ is cyclostationary of period T if and only if its Lamperti transform of parameter H: $\{X(t) = t^H Y(\ln t),\ t \in \mathbb{R}^+\}$, is (H, e^T)-DSI.

This is one possible extension of Lamperti's theorem, one of importance in our study of DSI. A first consequence, using (4), is that the general form of covariance of (H,λ)-DSI processes is naturally expressed on a Mellin basis:

$$R_X(t, kt) \;=\; k^H t^{2H} \sum_{n=-\infty}^{+\infty} C_n(k) t^{i2\pi n/\ln\lambda}. \qquad (7)$$

Note that if the process X is real-valued, a necessary condition is imposed: $C_{-n}(k) = C_n^*(k)$. The Mellin function $t^{H+i2\pi n/\ln\lambda}$ in (7) is central in the study of DSI processes. This is not a surprise: Lamperti transformation maps the Fourier basis (invariant up to a phase by time-shift) to the Mellin basis (invariant up to a phase by dilation and having also the deterministic DSI property). We stress the fact the Mellin functions are a basis and that they have an associated transformation which can be numerically computed [14].

3. EXAMPLES OF PROCESSES AND SEQUENCES WITH DSI

Continuous-time systems with DSI property are easily constructed. Applying $\mathcal{L}$ to an ARMA(p,q) system, we obtain a generalization of the Euler-Cauchy (EC) system. It is a model for self-similar processes [7], driven by a multiplicative Gaussian noise $\eta(t)$, whose correlation is $\mathbb{E}\{\eta(t)\eta(s)\} = t\sigma^2\delta(t-s)$. The process $X(t)$ verifies

$$\sum_{n=0}^{p} b_n t^n \frac{\mathrm{d}^n}{\mathrm{d}t^n} X(t) = \sum_{m=0}^{q} a_m t^{m+H} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \eta(t). \qquad (8)$$

In the same manner that a nonstationary ARMA models, with periodic time-varying coefficients is cyclostationary [15], one obtains a DSI model when taking log-periodic time-varying coefficients a_m and b_n in the (EC) system. This we will not detail further.

The synthesis of random sequences with DSI is more crucial because it is sequences that are simulated. Two properties are needed : self-similarity and log-periodicity. A possibility is to obtain a discrete-time system analog to (EC) (H-ss in a certain way), then introduce log-periodicity in the coefficients. We describe two approaches here.

A direct discretization in time of the (EC) system is given by the integration of its evolution between two instants. This was proposed in [16] for the first order. This nonstationary H-ss system is written as $X_k = a[k]X_k + e_k$, where $a[k] \simeq 1 - \alpha/k$ and e_k is a time-decorrelated Gaussian noise with variance $\mathbb{E}\, e_k^2 \propto k^{2H-1}$, when k is large. The generalization to the discretization of (EC) of order n is straightforward. The result is of the form, for the large times k

$$\begin{aligned}(1-B)^n X_k + a_1 k^{-1}(1-B)^{n-1} X_{k-1} \\ = \quad k^{-2}\mathrm{AR}(n-1)X_{k-1} + e_k + \mathcal{O}(k^{-3}) \quad (9)\end{aligned}$$

where B is the backward operator, and AR is an AR model.

Such a system with log-periodic coefficients a_1 and in the AR, or in cascade with a log-periodic AR system (see for example the AR(1) proposed hereafter, equation 11), will present an approximate DSI property. The reader can see on figure 1 on the left a realization of such a process.

Another class of discrete-time self-similar systems is given by models constructed on fractional difference operator [17, 18]. The usual method is to use its moving average representation written as a binomial expansion. We prefer to use the discretization proposed in [19], constructed with some generalization of the bilinear transformation in order to define a scaling operator for sequences. The fractional difference operator is then a filter $l_1[n]$ whose impulse response is:

$$l_1[n] = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(r+k)\Gamma(-r+n-k)}{\Gamma(k+1)\Gamma(n-k+1)\Gamma(r)\Gamma(-r)}. \qquad (10)$$

This filter is in cascade with a nonstationary MA filter whose coefficients are log-perodic. For example we may limit ourselves to the first order, taking care that the filter is stable at each instant:

$$l_2[1] = -\rho\left(1 + \beta_r \cos\left(\frac{2\pi \ln t}{\ln \lambda}\right)\right) e^{if_0(1+\beta_f \cos(2\pi \frac{\ln t}{\ln \lambda}))}. \qquad (11)$$

We propose an example of such a signal fig. 1 on the right. Note that this process is stationary where the previous model was not.

4. ANALYSIS BY DELAMPERTIZATION

Confronted to a general class of processes (or random sequences in the context of numerical processing) which are nonstationary, nor of a known structure, one has to find methods to analyse those. Given a sequence X_n suspected of DSI, the simplest way of analysis is to find the presumed cyclostationary process associated by applying $\mathcal{L}^{-1}$.

Generally speaking, classical stationary methods are usefull to analyse self-similar process after "delampertization" of the signal. This was the essence of papers on H-ss processes cited before [7, 8, 9]. Nonstationary methods can then be used tu study classes of processes which have not

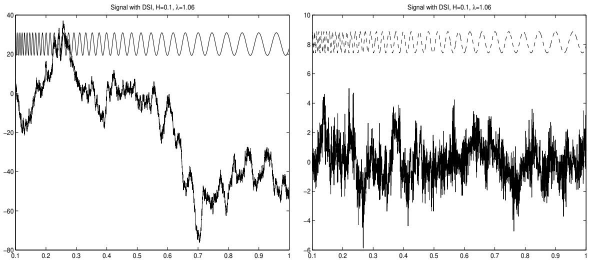

Fig. 1. Typical realizations of DSI random sequences. On the left the model is a (EC) system of order 2 discretized, in cascade with a log-periodic AR(1). On the right it is construted on fractional diffrence (see text). The length is 5000 points, $H = 0.1$, $\lambda = 1.0$. The oscillations above the signals are indicative of the log-periodicity of the AR. $f_0 = \pi/4$, $\rho = 0.6$, $\beta_r = 0.39$ and $\beta_f = 0$.

proper self-similarity, but which have some kind of nonstationarity with regards to dilation - a nonstationarity in scale. DSI is then only a first interesting exemple of a precise kind of nonstationarity in scale.

Before using cyclostationary methods, a practical problem must be considered : how compute in discrete time the inverse Lamperti transformation ? First, it needs a non linear sampling $t = q^n$ of the data but such is not often the case with real signals, or an interpolation to find the data with this geometrical sampling, given a signal X_t with usual arithmetic sampling: the corresponding sequence Y_t is known for $t = \ln n$, with $n \in \mathbb{N}$ and we want it for $t = m, m \in \mathbb{Z}$. Figure 2 shows on the left the sequence Y constructed from the second process on figure 1.

A second difficulty is that H is a priori unknown. Using the transformation of parameter H seems tricky... In fact the tools used thereafter are found not sensitive to this amplitude effect. The cyclostationary tools are found unaffected if one uses $H = 0.5$ to delampertize the process in place of the real H.

We tried the applicability of these ideas on synthetic sequences. As an example of classical cyclostationary tool, we implemented the methods proposed in [20]. In a nutshell the algorithm to estimate a time-smoothed cyclic cross periodogram is as follows. First the signal is decomposed in N segments of length L in order to average on these parts. A filtered and decomposed version is computed, where h is a data tapering window:

$$\tilde{Y}_T(n,f) = \sum_{l=-N/2}^{N/2} h(r)Y(n-l)e^{-i2\pi f(n-l)T_e} \qquad (12)$$

Then the spectral components $\tilde{Y}_T(n,\cdot)$ are correlated at frequencies $f - \nu_c/2$ and $f + \nu_c/2$ by a multiplier followed

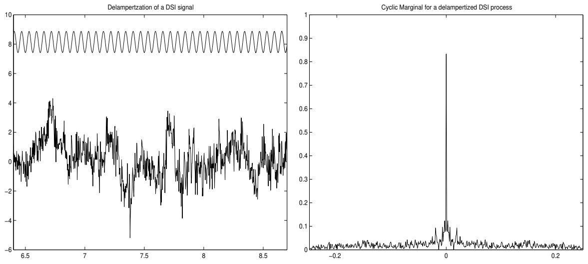

Fig. 2. On the left is shown the cyclostationary sequence after using $\mathcal{L}^{-1}$. On the right the marginal in cyclic frequency is reprensented. The main peak on the center in the total energy of the signal. The two symmetric peaks (pointed on by arrows) are an indication of cyclostationrity and situated to frequency $2\pi/\ln\lambda$.

by a low-pass filter g:

$$S_Y^{\nu_c}(v,f)=\sum_n \tilde{Y}_T(n,f-\frac{\nu_c}{2})\tilde{Y}_T^*(n,f+\frac{\nu_c}{2})g(v-n).$$

This is an estimate of the spectral cross correlation. It is related to the correlation (6) via two Fourier transform; theoretically it has the form:

$$S(f,\nu_c)=\sum_n \tilde{C}_n(f)\delta(\nu_c-\frac{n}{T}). \tag{13}$$

A reliable estimation of the cyclostationary period T, then of the prefered scale factor $\lambda=e^T$ is to compute the marginal in cyclic frequency of this spectrum. It has to have sharp peaks on $1/T$. See, figure 2, the result of this procedure for the synthetic model described before.

5. TOWARD MELLIN-BASED TOOLS

Another way of thinking might be fecund to analyse DSI processes. We can formulate directly the methods in a Mellin formalism, with no geometrical resampling. That is to say that we oper a "lampertization" of the tools where the first way proposed to "delampertize" the signal studied.

By direct interpolation we have few details for the short times (in fact we can't reconstitute $m<0$) and we ignore many details in the long times (taking one point among many). To obtain statistical relevance, one has to have a huge number of points in the original data to make some processing. The avantage, remarked in [8, 10], is that there are fewer points in X, then Y, after geometrical resampling and this keeps the computational cost low.

When one does not dipose of a large number of points, using a geometric sampling lose much information of the signal. As the Fourier transform of a process is related to the Mellin transform of the process transformed by $\mathcal{L}$, many methods for cyclostationary processes can be written with Mellin transformation and used on processes with DSI. For self-similar signals ($H=0$), estimators constructed in this way were given in [21]. Following the work of Hurd [22] on cyclostationarity, an estimator of $C_n(k)$ in the correlation function 7 for DSI, is given by

$$\hat{C}_n(A,k)=\frac{k^{-H}}{\ln A}\int_1^A X(kt)\bar{X}(t)t^{-in2\pi/\ln(\lambda)}\frac{\mathrm{d}\,t}{t^{2H+1}}. \tag{14}$$

This estimator is biased, $\mathbb{E}\hat{C}_n(A,k)=C_n(k)(1-|K|/A)+\mathcal{O}(A^{-1})$, and consistent if $\lim_{k\to\infty}R_x(t,kt)=0$ uniformly in t (see [22]).

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