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DISPLACEMENT RANK OF GENERALIZED INVERSES OF PERSYMMETRIC MATRICES*

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Abstract. Toeplitz matrices are persymmetric matrices belonging to the large class of so-called structured matrices, characterized by their displacement rank. This characterization was introduced 12 years ago by Kailath and others. In this framework, properties of singular structured persymmetric matrices are investigated with the goal of proving the possible existence of fast algorithms for computing their pseudo-inverses. Loosely speaking, it is proved that the pseudo-inverses of some structured matrices with displacement rank r have a displacement rank bounded by $2r$.

Key words. Toeplitz, structure, persymmetric, singular, Schur algorithm, pseudo-inverse

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1. Introduction. Toeplitz matrices and more generally structured matrices are encountered in several problems, including prediction of almost stationary processes, modeling of stochastic processes by state-space systems, lossless transmission lines, localization in antenna array processing, or testing relative primeness of polynomials [1], [2], [5], [6]. Because of their sometimes very large size, the structured linear systems must be solved by resorting to specialized algorithms taking advantage of their particular features in order to reduce both computational load and storage requirements, without disregarding the possibilities of parallel implementations.

There exist numerous fast algorithms for solving linear Toeplitz systems, and the most well known are the Levinson and Schur algorithms. Extensions have been proposed since 1979 for matrices whose structure was close to Toeplitz in some sense [2], [3]. For full-rank matrices, the main results may be found in [3]–[5] and [9]. Nevertheless, the leading minors are required to be nonzero in order for existing fast algorithms to be stable [2]. Improvements have been proposed in order to allow fast algorithms to run in a stable way for the larger class of regular matrices with arbitrary rank profile [12], [14]. Besides this first limitation, it is now proved that the proximity of a regular matrix to the Toeplitz structure is preserved by inversion, but nothing is known to date regarding singular matrices. Singular structured matrices will be the subject of our discussion, and we shall focus our attention on persymmetric matrices (relevant in the Toeplitz case, for instance). It will be proved that the generalized inverse of a singular structured persymmetric matrix has a structure that could be defined with the help of generators, exactly as in the regular case [3]; in other words, its displacement rank is bounded. Our approach, however, is not constructive in the sense that no means is provided to obtain explicitly the corresponding generators. Only their existence is proved. This result is important since it demonstrates that the generalized inverse of an $N \times N$ close to Toeplitz matrix can be completely described only by a restricted number of vectors of size N (four in the case of a singular Toeplitz matrix).

The paper is organized as follows. Section 2 defines notations and states mostly known results for regular matrices. The body of the paper is § 3, in which properties of some singular symmetric persymmetric structured matrices are investigated. Section 4

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briefly extends the result of § 3 to the nonsymmetric case. Although Theorem 3.1 is a particular case of Theorem 4.1 (§ 3 could be partially skipped), we found it clearer to go first through the less general symmetric case.

2. Displacement of matrices. The concept of matrix displacement has been introduced in [2], and discussed in a more general framework in [5]. Only a few definitions and notations are recalled below for convenience, and readers are invited to consult the above-mentioned references as well as [1], [3], and [4] for more details.

DEFINITION 2.1. Let Z be a fixed nilpotent square matrix. Two types of displaced matrices will be used; for any square matrix T, they are defined as

$$\nabla T = T - ZTZ \quad \text{and} \quad \Delta T = T - Z^t TZ. \tag{2.1}$$

The rank of matrix ∇T (respectively, ΔT) is called the displacement rank of T with respect to Z (respectively, Z^t).

DEFINITION 2.2. Any set of pairs of vectors $\{(x_1, y_1), (x_2, y_2), \dots, (x_q, y_q)\}$, satisfying

$$\nabla T = \sum_{i=1}^{q} \alpha_i x_i y_i^t, \qquad \alpha_i \in \{-1, 1\}, \tag{2.2}$$

is a set of generators. If the matrix T is symmetric, then we can take $x_i = y_i$ and the sequence of signs $\{\alpha_i\}$ is called the displacement signature of T.

If q is minimum, i.e., if q is equal to the displacement rank of T, then those generators are called minimal [7]. Only minimal generators will be referred to in the rest of the paper. The Crout decomposition of ∇T is one way for building minimal generators.

Property 2.1. As indicated by their name, generators can be used to recover the original matrix. In fact, it suffices to form the sum

$$T = \sum_{i=0}^{N-1} Z^i \nabla T Z^{it}. \tag{2.3}$$

Another equivalent expression may be obtained by stacking the successive shifted generators in triangular matrices,

$$T = \sum_{i=1}^{q} \alpha_i L_i U_i, \quad L_i = [x_i \, Zx_i \cdots Z^{N-1} x_i], \quad U_i = [y_i \, Zy_i \cdots Z^{N-1} y_i]^t. \tag{2.4}$$

This property shows that if a matrix T has a small displacement rank compared to its size, then it may be stored efficiently under the form of $2q$ generators (q are sufficient if T is symmetric).

DEFINITION 2.3. For the sake of simplicity, the displacements of the inverse matrix T^{-1} will be denoted in short as ∇T^{-1} and ΔT^{-1}, standing for $\nabla(T^{-1})$ and $\Delta(T^{-1})$. Inverses of displaced matrices are not used in the paper, thus avoiding any confusion.

THEOREM 2.1. *If T is a nonsingular square symmetric matrix, displaced matrices ∇T and ΔT^{-1} have the same rank. The same property holds for ΔT and ∇T^{-1}.*

This result is attributed to Gohberg and Semencul and can be traced back to 1972. The first proof given below can be found in [1], [4], and [5], and has some interest because of its conciseness. Useful relations with orthogonal polynomials are also stressed in [10], [11]. We derive then a second proof, more convenient for further extensions to the case of singular matrices. This emphasizes the differences between the principles utilized.

First proof. The shortest proof that can be given consists of writing the matrix

$$\begin{pmatrix} T & Z \\ Z^t & T^{-1} \end{pmatrix}$$

in two different manners:

$$\begin{pmatrix} I & ZT \\ 0 & I \end{pmatrix}\begin{pmatrix} \nabla T & 0 \\ 0 & T^{-1} \end{pmatrix}\begin{pmatrix} I & 0 \\ TZ^t & I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I & 0 \\ Z^tT^{-1} & I \end{pmatrix}\begin{pmatrix} T & 0 \\ 0 & \Delta T^{-1} \end{pmatrix}\begin{pmatrix} I & T^{-1}Z \\ 0 & I \end{pmatrix}.$$

Yet, from the Sylvester lemma, inertia is preserved by congruent transformations, which shows that ΔT^{-1} and ∇T have the same inertia, since so do T^{-1} and T. Note that this result is stronger than the theorem.

Second proof. We shall prove in three steps that the null spaces of ΔT^{-1} and ∇T have the same dimension. Denote $U = \text{Ker } \nabla T$ and $V = TZ^t(U)$ the image of U by the operator TZ^t. We prove first that

$$V \subset \text{Ker } \Delta T^{-1}. \tag{2.5}$$

Let $u \in U$ and $v = TZ^tu$. Then

$$\Delta T^{-1}v = T^{-1}TZ^tu - Z^tT^{-1}ZTZ^tu.$$

But since $u \in \text{Ker } \nabla T$,

$$ZTZ^tu = Tu,$$

and hence

$$\Delta T^{-1}v = Z^tu - Z^tT^{-1}Tu = 0,$$

which proves (2.5). Next TZ^t is one-to-one on U. In fact, if $TZ^tu = 0$ for $u \in U$, then $Tu = ZTZ^tu = 0$, implying $u = 0$ because T is nonsingular.

This implies that $\dim U = \dim V$, which together with (2.5) and the definition of U yields

$$\dim \nabla T \leqq \dim \text{Ker } \Delta T^{-1}.$$

A similar argument with T and Z^t replaced by T^{-1} and Z, and with ∇ and Δ interchanged, implies the reverse inclusion. This completes the proof. □

THEOREM 2.2. *Let the matrix Z denote the so-called "lower shift" matrix*

$$Z = \begin{pmatrix} \cdot\cdot & 0 & : & 0 \\ & \cdot\cdot & : & \cdot\cdot \\ & I_{N-1} & : & 0 \\ & & : & \end{pmatrix}.$$

Then a symmetric Toeplitz matrix T always has a displacement rank 2 with respect to Z. Moreover, if diagonal entries of T are normalized to 1, the range of ∇T is spanned by t_1 and ZZ^tt_1, where t_1 denotes the first column of T. The vectors t_1 and ZZ^tt_1 are minimal generators of ∇T.

Proof. See [4] or [7] for a proof.

3. Displacement rank of Moore–Penrose inverses.

DEFINITION 3.1. Define the generalized inverse of any square matrix T, denoted T^-, as the matrix satisfying the four Moore–Penrose conditions: (i) $TT^-T = T$, (ii) $T^-TT^- = T^-$, (iii) $(TT^-)^t = TT^-$, and (iv) $(T^-T)^t = T^-T$.

The null space of the generalized inverse T^- is the null space of T^t and the range of T^- is the range of T^t [8]. Hence in the symmetric case the generalized inverse T^- has the same range and null spaces as T.

For the sake of simplicity, we shall only concern ourselves with the symmetric case in this section. The nonsymmetric case will be postponed to § 4.

DEFINITION 3.2. Define the backward identity matrix J as

$$J_{i,j} = 0 \quad \text{except } J_{i,N-i+1} = 1.$$

J is sometimes called the anti-identity matrix, or the reverse unit matrix. A matrix M will be called persymmetric if it satisfies

$$JMJ = M^t.$$

From now on the displacement matrix Z will be assumed to be persymmetric.

THEOREM 3.1. *Let T be an $N \times N$ symmetric and persymmetric matrix. Then if r is the displacement rank of T, the displacement rank of its generalized inverse T^-, is bounded by $2r$.*

Let us examine step by step the proof derived above in the nonsingular case for T^{-1}. The first obstacle in extending it to the generalized inverse of T is that we cannot use the property $T^{-1}T = I_N$, but only $TT^-T = T$; this means that the main step of the proof fails, namely, that $TZ^t(\text{Ker } \nabla T) \subset \text{Ker } \Delta T^{-1}$.

Nevertheless, we can still prove, as we shall see, that the quadratic form associated with ΔT^- (and not the linear operator itself) vanishes on a subspace E of dimension $N - r$; from this it follows that the dimension of Ker ΔT^- is smaller than $N - 2r$, as will be shown subsequently. This subspace E is built as a sum of two subspaces, $V \oplus W$, where $V = TZ^t(\text{Ker } \nabla T)$ and W is a subspace of Ker T. The proof of Theorem 3.1 requires two lemmas.

LEMMA 3.1. *The quadratic form $\langle \Delta T^- v, v \rangle$ vanishes for all $v \in V$, $V = TZ^t(\text{Ker } \nabla T)$.*

Proof. Let u be in Ker ∇T and $v = TZ^t u$. Let us calculate $\langle \Delta T^- v, v \rangle$:

$$\langle \Delta T^- v, v \rangle = \langle T^- TZ^t u, TZ^t u \rangle - \langle Z^t T^- Z T Z^t u, TZ^t u \rangle.$$

By using the transpose rule for operators and the symmetry of T, we get

$$\langle \Delta T^- v, v \rangle = \langle TT^- TZ^t u, Z^t u \rangle - \langle T^- Z T Z^t u, ZTZ^t u \rangle.$$

But $ZTZ^t u = Tu$ for $u \in \text{Ker } \nabla T$, and $TT^-T = T$ by definition of T^-. Hence

$$\langle \Delta T^- v, v \rangle = \langle TZ^t u, Z^t u \rangle - \langle T^- Tu, Tu \rangle,$$

and resorting to transposition gives

$$\langle \Delta T^- v, v \rangle = \langle ZTZ^t u, u \rangle - \langle TT^- Tu, u \rangle.$$

Now using the properties $ZTZ^t u = Tu$ and $TT^-T = T$ again yields

$$\langle \Delta T^- v, v \rangle = \langle Tu, u \rangle - \langle Tu, u \rangle = 0. \qquad \square$$

LEMMA 3.2. *Define K as the null space of the operator TZ^t in* Ker ∇T, *i.e.*,

$$K = \text{Ker } TZ^t \cap \text{Ker } \nabla T.$$

Define also the subspace

$$W = J(K).$$

With the notations defined above, we have

(i) $W \subset \text{Ker}\ \Delta T^-$,

(ii) $\dim(W) = \dim(K)$, *and*

(iii) $W \subset \text{Ker}\ T$.

Proof. Symmetry and persymmetry imply centrosymmetry:

$$T = JTJ,$$

while the following identities are also satisfied:

$$Z = JZ^tJ \quad \text{and} \quad J^2 = I_N.$$

First, let us check that W is included in Ker ΔT^-. Let x be in K and note that this is equivalent to

$$TZ^tx = 0 \quad \text{and} \quad Tx = ZTZ^tx.$$

Thus $x \in K$ if and only if

$$TZ^tx = 0 \quad \text{and} \quad Tx = 0. \tag{3.1}$$

Define $y = Jx$ and notice that $x = Jy$. Thus

$$Ty = JTJy = JTx = 0, \tag{3.2}$$

$$TZy = JTJJZ^tJy = JTZ^tx = 0. \tag{3.3}$$

Now it follows from the definition of the generalized inverse T^- that matrices T and T^- have the same null space. Hence from (3.2),

$$T^-y = 0,$$

and from (3.3),

$$T^-Zy = 0.$$

These two results prove statement (i) of Lemma 3.2, namely,

$$W \subset \text{Ker}\ \Delta T^-.$$

Next, K and $W = J(K)$ have the same dimension since J is one to one. This proves (ii). Finally, result (3.2) immediately gives us the assertion (iii). □

Proof of Theorem 3.1. On one hand, the quadratic form associated with the matrix ΔT^- is null on the subspace V. On the other hand, the subspace W is included in the null space of the linear operator ΔT^-. From these properties, it is straightforward to conclude that the quadratic form is null on the whole subspace $E = V + W$. Moreover, $V \perp W$ because $V \subset$ range T and $W \subset \text{Ker}\ T$, hence $E = V \oplus W$ and

$$\dim(E) = \dim(V) + \dim(W),$$

$$\dim(E) = \dim(V) + \dim(K).$$

The dimension of $V = TZ^t(\text{Ker}\ \nabla T)$ is obtained from the dimension rule for the range and null space of the restriction of TZ^t to Ker ∇T:

$$\dim(V) = \dim(\text{Ker}\ \nabla T) - \dim(K). \tag{3.4}$$

Thus

$$\dim(E) = \dim(\text{Ker}\ \nabla T) = N - r.$$

The quadratic form $\langle \Delta T^{-}x, x\rangle$ vanishes at least on a subspace E of dimension $N-r$. Consider an orthogonal basis whose first $N-r$ vectors form a basis of E. In such a basis this quadratic form is defined by a matrix $P\Delta T^{-}P^{t}$ that has at most r nonzero rows and r nonzero columns, as shown in the following:

$$P\Delta T^{-}P^{t} = \begin{pmatrix} & & & & & X \\ & & & & & X \\ & & 0 & & & X \\ & & & & & X \\ & & & & & X \\ X & X & X & X & X & X \end{pmatrix} \}\, r \text{ rows}.$$

Thus its rank is at most $2r$, and consequently the rank of ΔT^{-} is at most $2r$. □

Notice that Toeplitz matrices are persymmetric, and we therefore have Theorem 3.2.

THEOREM 3.2. *Let T be an $N \times N$ symmetric Toeplitz matrix, and Z be given as in Definition* 3.2. *Then the displacement rank of its generalized inverse T^{-} is bounded by* 4.

Remark 3.1. In order to generate a positive Toeplitz matrix of rank r for numerical experiments, we use a form of Caratheodory's representation [13], constructed with the help of Krylov subspaces. We first build an arbitrary orthogonal matrix Q with an invariant subspace of dimension N-r, as a product of r arbitrarily chosen symmetries. Then, we generate a vector x at random and we form the matrix M, whose columns are the vectors $x, Qx, \ldots, Q^{N-1}x$. Notice that the Krylov subspace built with matrix Q and starting vector x is indeed of dimension r. The required matrix is the covariance matrix $T = M^{t}M$.

Remark 3.2. As particular cases, let us insist that generalized inverses of $N \times N$ symmetric Toeplitz matrices of rank 1 have in general a displacement rank of 2. If their rank is larger than 1 and smaller than N, they will have in general a displacement rank of 4 from Theorem 3.2. In particular cases, however, their displacement rank may fall to 3 (see the example below).

Example. This simple example illustrates the practical issues addressed in the remarks above. Let $N = 5$ and $r = 3$, and define the vectors $a = (1\ 0\ 0\ 0\ 0)^{t}$, $b = (1/\sqrt{2}\ 1/\sqrt{2}\ 0\ 0\ 0)^{t}$, and $c = (0\ 1/\sqrt{2}\ 1/\sqrt{2}\ 0\ 0)^{t}$. Following the procedure proposed in Remark 3.1, the orthogonal matrix $Q = (I - 2bb^{t})(I - 2cc^{t})$ is built, and the matrix $M = [a\ Qa\ Q^{2}a\ Q^{3}a\ Q^{4}a]$,

$$Q = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now, a symmetric Toeplitz matrix of rank 3 can be obtained by computing $M^{t}M$. We get

$$T = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \qquad 4T^{-} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 4 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

It can be checked that the displacement rank of T^- is 3. Another example would show that the bound can be reached for the same value of rank (T). Change a into $(0\ 1\ 0\ 1\ 0)^t$, for instance, and get a displacement rank of 4 for T^-, with

$$T = \begin{pmatrix} 2 & 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 & 2 \end{pmatrix}, \qquad 16T^- = \begin{pmatrix} 3 & -1 & -2 & 3 & -1 \\ -1 & 3 & -2 & -1 & 3 \\ -2 & -2 & 12 & -2 & -2 \\ 3 & -1 & -2 & 3 & -1 \\ -1 & 3 & -2 & -1 & 3 \end{pmatrix}.$$

4. Nonsymmetric case. In this section we extend Theorem 3.1 to the wider class of persymmetric matrices, containing in particular the nonsymmetric Toeplitz matrices.

THEOREM 4.1. *Let T be an $N \times N$ persymmetric matrix with displacement rank r. Then the displacement rank of its generalized inverse T^- is bounded by* $2r$.

First we need to modify Lemma 3.1.

LEMMA 4.1. *The subspaces $V_1 = TZ^t(\text{Ker } \nabla T)$ and $V_2 = T^tZ^t(\text{Ker } \nabla T^t)$ are orthogonal for the bilinear form $\langle \Delta T^- y, z \rangle$.*

Proof. The proof is derived in a similar manner as in Lemma 3.1. Let $v_1 = TZ^tu_1$ with u_1 in Ker ∇T and let $v_2 = T^tZ^tu_2$ with u_2 in Ker ∇T^t. Now calculate $\langle \Delta T^- v_1, v_2 \rangle$,

$$\langle \Delta T^- v_1, v_2 \rangle = \langle T^-TZ^tu_1, T^tZ^tu_2 \rangle - \langle Z^tT^-ZTZ^tu_1, T^tZ^tu_2 \rangle.$$

By using the transpose rule for operators, we get

$$\langle \Delta T^- v_1, v_2 \rangle = \langle TT^-TZ^tu_1, Z^tu_2 \rangle - \langle T^-Tu_1, ZT^tZ^tu_2 \rangle.$$

But $ZT^tZ^tu_2 = T^tu_2$ for $u_2 \in \text{Ker } \nabla T^t$, and $TT^-T = T$ by definition of T^-. Hence

$$\langle \Delta T^- v_1, v_2 \rangle = \langle TZ^tu_1, Z^tu_2 \rangle - \langle T^-Tu_1, T^tu_2 \rangle,$$

and resorting to transposition gives

$$\langle \Delta T^- v_1, v_2 \rangle = \langle ZTZ^tu_1, u_2 \rangle - \langle TT^-Tu_1, u_2 \rangle.$$

Another use of the properties $ZTZ^tu_1 = Tu_1$ for $u_1 \in \text{Ker } \nabla T$ and $TT^-T = T$ yields

$$\langle \Delta T^- v_1, v_2 \rangle = \langle Tu_1, u_2 \rangle - \langle Tu_1, u_2 \rangle = 0. \qquad \square$$

Next we modify Lemma 3.2.

DEFINITION 4.1. Define K_1 as the null space of the operator TZ^t in Ker ∇T, i.e.,

$$K_1 = \text{Ker } TZ^t \cap \text{Ker } \nabla T,$$

and define K_2 as the null space of the operator T^tZ^t in Ker ∇T^t, i.e.,

$$K_2 = \text{Ker } T^tZ^t \cap \text{Ker } \nabla T^t.$$

Also define W_i as the subspace $J(K_i)$.

LEMMA 4.2. *The subspace W_1 is included in the null space of the linear operator ΔT^-, and W_2 is included in the null space of the linear operator ΔT^{t-}. Moreover,*

$$\dim (W_i) = \dim (K_i),$$

$$W_1 \subset \text{Ker } T^t \quad \textit{and} \quad W_2 \subset \text{Ker } T.$$

Proof. Recall that $T^t = JTJ$, $Z = JZ^tJ$, and $J^2 = I_N$. First, let us check that W_1 is included in Ker ΔT^-. Let x be in K_1 and remark that this is equivalent to

$$TZ^tx = 0 \quad \text{and} \quad Tx = ZTZ^tx.$$

Thus $x \in K_1$ if and only if

$$TZ^t x = 0 \quad \text{and} \quad Tx = 0. \tag{4.1}$$

Define $y = Jx$. Noticing that $x = Jy$ gives

$$T^t y = JTJy = JTx = 0, \tag{4.2}$$

$$T^t Zy = JTJJZ^t Jy = JTZ^t x = 0. \tag{4.3}$$

Now by definition of the generalized inverse T^-, the operators T^t and T^- have the same null space. Hence

$$T^- y = 0 \quad \text{and} \quad T^- Zy = 0.$$

This proves that $W_1 \subset \operatorname{Ker} \Delta T^-$. Next, J is one to one; thus K_1 and $W_1 = J(K_1)$ have the same dimension, whereas the result (4.2) gives us $W_1 \subset \operatorname{Ker} T^t$. A similar proof holds for W_2. □

Proof of Theorem 4.1. The subspaces V_1 and V_2 are orthogonal with respect to the bilinear form associated with the matrix ΔT^-. The subspace W_1 is included in the null space of the linear operator ΔT^- while the subspace W_2 is included in the null space of the linear operator ΔT^{t-}. From these properties, it is straightforward to conclude that the whole subspaces $E_1 = V_1 + W_1$ and $E_2 = V_2 + W_2$ are orthogonal, according to the bilinear form associated with the matrix ΔT^-. Indeed, if y_1, z_1, y_2, and z_2 are, respectively, in V_1, W_1, V_2, and W_2,

$$\langle \Delta T^-(y_1 + z_1), (y_2 + z_2) \rangle = \langle \Delta T^- y_1, y_2 \rangle + \langle \Delta T^- z_1, (y_2 + z_2) \rangle + \langle y_1, \Delta T^{t-} z_2 \rangle,$$

and each term on the right-hand side of this expression is null from preceding results.

Moreover, $V_1 \perp W_1$ because $V_1 \subset \operatorname{range} T$ and $W_1 \subset \operatorname{Ker} T^t$. Hence $E_1 = V_1 \oplus W_1$ and, following the proof of the symmetric case, we may conclude that

$$\dim (E_1) = N - r.$$

The bilinear form $\langle \Delta T^- x, y \rangle$ has two orthogonal subspaces E_1 and E_2, both of dimension at least $N - r$. It is easy to conclude, as in the symmetric case, that the rank of this bilinear form is at most $2r$ and consequently the rank of matrix ΔT^- is at most $2r$. □

5. Concluding remarks. Theorem 4.1 says that if a persymmetric matrix T has a displacement rank r with respect to a persymmetric displacement matrix Z, then its pseudo-inverse T^- has a displacement rank bounded by $2r$. In other words, the matrices T and T^- can be completely characterized by at most $2r$ and $4r$ generating vectors, respectively. If T is symmetric Toeplitz, four vectors are sufficient to characterize matrix T^-. Now in order for this result to be fully exploited, it would be necessary to find an algorithm able to express explicitly the generators of T^-. This issue is left open.

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