Generic and Typical Ranks of Multi-Way Arrays

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Abstract

The concept of tensor rank was introduced in the twenties. In the seventies, when methods of Component Analysis on arrays with more than two indices became popular, tensor rank became a much studied topic. The generic rank may be seen as an upper bound to the number of factors that are needed to construct a random tensor. We explain in this paper how to obtain numerically in the complex field the generic rank of tensors of arbitrary dimensions, based on Terracini's lemma, and compare it with the algebraic results already known in the real or complex fields. In particular, we examine the cases of symmetric tensors, tensors with symmetric matrix slices, complex tensors enjoying Hermitian symmetries, or merely tensors with free entries.

Key words: Tensor, Multiway Arrays, Generic Rank, Canonical Decomposition, Parafac, Factor Analysis

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1 Introduction

Generic ranks, defined in the complex field, have been studied for several decades [1] [2]. However, the value of the generic rank for arbitrary dimensions is not yet known in the unsymmetric case, and has been known in the symmetric case only recently [3] [4] [5]. The existence itself of the generic rank is not ensured in the real case, and there exist in general several *typical ranks* (see section 2.1 for definitions). The typical tensor rank of three-way arrays over algebraically closed fields has been much studied in the context of computational complexity theory. Bürgisser, Clausen and Shokrollahi [6, Ch.20] give an overview of general results for various classes of arrays; these results have been extended in [5] in the symmetric case. The study of tensor rank over the real field has lagged behind. In this paper, generic and typical ranks are discussed for various tensor structures.

The typical rank of three-way arrays over the real field has been relevant for psychological data analysis since Carroll and Chang [7] and Harshman [8] independently proposed a method which they christened CANDECOMP and PARAFAC, respectively. Therefore, we shall subsequently refer to this decomposition with the acronym CP, as several other authors did before, even if the CP decomposition had been introduced much earlier [9]. This CP decomposition generalizes Principal Component Analysis to three-way data, by seeking the best least squares approximation of a data array by the sum of a limited number of rank-one arrays. In 2-way analysis, the rank of the data matrix is the maximum number of components that Principal Component Analysis can extract, up to scale and rotation ambiguities. This property generalizes smoothly to three-way data. That is, the rank of a three-way array is the maximum number of components that CP can extract, up to scale and permutation ambiguities. Thus, the study of typical rank of three-way arrays is of great theoretical importance for CP.

Although CP was developed in a psychometric environment, its main area of applications has been Chemometrics, e.g. [10]. In addition to straightforward

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application of CP, chemometricians also use Tucker3 component analysis [11] [12] [13] quite often. This is another model aiming at decomposing a data array as weighted sum of rank-one arrays, the weights being collected in a so-called core array. Typically, the underlying chemometric model dictates that a vast majority of specified core elements vanish. Because there exist admissible transformations which generate a vast majority of zero elements in arbitrary arrays, we need tools to tell models from tautologies. This is where the concept of typical rank has found another realm of application. For instance, Ten Berge and Smilde [14] have argued that a sparse core hypothesized by Gurden et al. [15] is indeed a model and not a tautology. Their hypothetical core was a $5\times5\times3$ array with only 5 nonzero entries, hence of rank 5 at most. Because $5\times5\times3$ arrays have a typical rank of at least 7, it is clear at once that transformations which yield as few as five non-zero elements, starting from any randomly generated $5\times5\times3$ array, do not exist. In this way, the typical rank of three-way arrays finds applications in distinguishing constrained Tucker3 models from tautologies. Besides Chemometrics, CP has found important applications in signal processing, especially in Independent Component Analysis [16] [17] and in multi-user access in wireless communications [18] [19] [20]. Moreover, the decomposition is finding its way to scientific computing, where it leads to a way around the Curse of Dimensionality [21, p. 125] [22] [23].

The paper is organized as follows. In section 2, definitions and historical remarks are provided. Next, a numerical algorithm is described in section 3, which is able to compute the generic rank of any tensor, symmetric or not. This approach is based on the so-called Terracini's lemma. Numerical values are reported in section 4, and compared to the already known rank values previously obtained by means of algebraic calculations. The consistency of the results confirm the validity of the approach, which can yield generic ranks for more complicated structures, such as tensors with symmetric matrix slices, which occur in the context of the INDSCAL model [7], among others.

2 Generic and Typical Ranks

2.1 Definitions

Let $\boldsymbol{T}$ be a L-way array of dimensions N_ℓ, $1 \leq \ell \leq L$, with values in a ring $\mathcal{R}$. This array always admits a decomposition into a sum of outer products as:

$$\boldsymbol{T} = \sum_{r=1}^{R} \boldsymbol{u}_r^{(1)} \circ \boldsymbol{u}_r^{(2)} \circ \ldots \circ \boldsymbol{u}_r^{(L)} \tag{1}$$

where $\boldsymbol{u}_r^{(\ell)}$ is a vector of dimension N_ℓ, $\forall r$, and $\circ$ denotes the tensor product [2].

Now consider an array $\boldsymbol{T}$ with values in a field $\mathbb{K}$. Arrays $\boldsymbol{u}_r^{(\ell)}$ may be considered as vectors of the linear space $\mathbb{K}^{N_\ell}$. Thus, as a combination of tensor products of vectors, $\boldsymbol{T}$ may be considered as a tensor. Under a linear change of coordinate system in each space $\mathbb{K}^{N_\ell}$, defined by a matrix $\boldsymbol{A}^{(\ell)}$, the tensor is represented by another array, obtained by the multi-linear transform $\{\boldsymbol{A}^{(1)}, \boldsymbol{A}^{(2)}, \ldots, \boldsymbol{A}^{(L)}\}$. Since it is legitimate once a basis has been defined in the space, no distinction will be made in the remainder between the tensor and its array representation.

The *rank* of a given tensor $\boldsymbol{T}$ (and by extension, of the array defining its coordinates in a given basis) is the minimal integer R such that the decomposition (1) is exactly satisfied. This decomposition is referred to as the tensor *Canonical* or *Parallel factor* Decomposition (CP).

A property is called *typical* if it holds true on a set of nonzero volume [2] [24] [25] [5]. This supposes that some topology has been defined on $\mathbb{K}^{N_1 \times N_2 \times \ldots N_L}$; this can be the Zariski topology for instance, or an Euclidean topology. A property is said to be *generic* if it is true almost everywhere. In other words, a generic property is typical, but the converse is not true.

Let $N_1, \ldots, N_L$ be given positive integers. Then the rank of tensors of size $N_1 \times N_2 \times \cdots \times N_L$ is bounded, and one can make a partition of the tensor space, according to the rank values. One can define *typical ranks* as the ranks that are associated with subsets of nonzero volume in the latter partition. If there is a single typical rank, then it may be called the *generic rank*.

For instance, there is a single generic rank if the underlying field $\mathbb{K}$ is algebraically closed (as the field of complex numbers, $\mathbb{C}$) [2] [5]. But there may be several typical ranks if $\mathbb{K}$ is the real field, $\mathbb{R}$. In the complex field, the calculation of the generic rank of symmetric tensors is completely described by the Alexander-Hirschowitz (AH) theorem [3]. Recently, Abro and Ottaviani attempted to generalize the AH theorem to non symmetric complex tensors [26], and provided an almost exhaustive list of exceptions. This recent contribution is the most significant step towards the complete characterization of the generic rank of unsymmetric tensors with free entries.

[2] This notation is used in order to make the distinction with the matrix Kronecker product used in the next section.

2.2 Historical remarks

Bounds on the typical rank over the complex field were given in [2]. The study of typical rank over the real field was initiated by Kruskal [27] [24], who noted that $2 \times 2 \times 2$ arrays had both rank 2 and rank 3 with positive probability. Kruskal also added a few typical ranks for small arrays. Franc [28] discussed some more results, including bounds on typical rank. Ten Berge and Kiers [25] gave a first result of some generality, in solving the typical rank issue for all two-slice arrays (that is, arrays of format $2 \times N_2 \times N_3$). These results were further generalized in [29], to include all cases where, for $N_1 \geq N_2 \geq N_3$, $N_1 > N_2N_3 - N_2$. Additional miscellaneous results can be found in [30] [29] [31] [32].

When Carroll and Chang developed CANDECOMP, the main applications they had in mind (a scalar product fitting problem related to INDSCAL) involved three-way arrays with slices that are symmetric in two of the three modes. Ten Berge, Sidiropoulos and Rocci [33] noted that this form of symmetry would affect typical ranks, and examined a number of cases; also see [34]. Quite often, indeed, symmetry of slices appears to entail lower typical rank values. On the other hand, there are also cases where symmetry of the slices does not affect the typical rank. A partial explanation for this can be found in [32]. Ten Berge et al. also noted that symmetric slices are often double centered [7, p. 286], which will further reduce the typical rank. That is, when an array has N_1 double centered slices of order $N_2 \times N_2$, it can be reduced to a $N_1 \times (N_2 - 1) \times (N_2 - 1)$ array, and its typical rank will therefore be the same as that of noncentered symmetric-slice $N_1 \times (N_2 - 1) \times (N_2 - 1)$ arrays. A parallel reasoning can be carried out for any nonsymmetrical double centered slices. A rationale for double centering slices in the PARAFAC context can be found in [35, p. 239]. It is easy to show that the $N_1 \times N_2 \times N_3$ array with N_1 double centered slices has the same typical rank as the uncentered $N_1 \times (N_2 - 1) \times (N_3 - 1)$ array.

3 Computation of Generic Ranks

The algorithm proposed is directly inspired by [4], which is in turn based on the so-called Terracini's lemma [36] [37] [38]; note that the latter is often attributed to Lasker, and is hence almost one hundred years old. In a few works, the principle is based on the fact that the dimension of an irreducible variety is equal almost everywhere to the dimension of its tangent space (and we know that the set of tensors of rank at most R is irreducible [5], for any R). Thus the dimension of a variety can be computed by measuring the dimension

of the tangent space at a *generic* [3] point. Let's see now how to compute the tangent space of interest.

Equation (1) can be seen as a parameterization of tensor $\boldsymbol{T}$. In fact, given a set of vectors $\{\boldsymbol{u}_r^{(\ell)} \in \mathbb{K}^{N_\ell},\, 1 \leq \ell \leq L,\, 1 \leq r\}$, consider the mapping φ defined from a known subspace $\mathcal{T}_R$ of $(\mathbb{K}^{N_1} \times \mathbb{K}^{N_2} \times \cdots \times \mathbb{K}^{N_L})^R$ onto $\mathbb{K}^{N_1 N_2 \ldots N_L}$ as:

$$\{\boldsymbol{u}_r^{(\ell)} \in \mathcal{T}_R,\, 1 \leq \ell \leq L,\, 1 \leq r \leq R\} \rightarrow \sum_{r=1}^{R} \boldsymbol{u}_r^{(1)} \circ \boldsymbol{u}_r^{(2)} \circ \ldots \circ \boldsymbol{u}_r^{(L)}$$

Denote $\mathcal{Z}_R = \varphi(\mathcal{T}_R)$ the image of this mapping. Then the dimension D of its closure $\bar{\mathcal{Z}}_R$ is given by the rank of the Jacobian of φ taken at a generic point, expressed in any fixed basis of $\mathbb{K}^{N_1 N_2 \ldots N_L}$. If the Jacobian is of maximal rank at a generic point, that is, if its rank equals almost everywhere the dimension of the image space (e.g. $N_1 N_2 \ldots N_L$ for unconstrained arrays), then it means that R is a typical rank. Actually, R will be either the smallest typical rank, or the generic rank. Note that it is always possible to reach the maximal Jacobian rank by increasing the number of terms R, so that the smallest [4] typical rank is always found.

This result yields the following numerical algorithm:

- Express formally the parameterized rank-one tensor term in a canonical basis
- Express formally the gradient of the latter in this basis
- Draw randomly the parameters according to an absolutely continuous distribution, and initialize matrix $\boldsymbol{J}$ with the numerical value of the gradient
- $R = 1$
- While $\text{rank}(\boldsymbol{J})$ strictly increases, do:
  - Draw randomly the parameters according to an absolutely continuous distribution, and append this new numerical value of the gradient as a new row block in $\boldsymbol{J}$
  - Compute the new value of $D = \text{rank}(\boldsymbol{J})$
  - $R \leftarrow R + 1$
- Compute the dimension of the fiber of solutions as $F = M - D$, the difference between the number of parameters and the dimension of the image $\bar{\mathcal{Z}}_R$.

In order to clarify the description of this algorithm, we give now the exact expressions of the Jacobian in various cases.

[3] In practice, a generic point is drawn randomly. If the result is questionable, the point can be drawn another time. Any random drawing should yield the same dimension, if the pdf is absolutely continuous.

[4] Finding all typical ranks is still an open problem, and is not addressed in this paper.

3.1 Jacobian for 3rd order asymmetric tensors with free entries

The mapping takes the form below

$$\{\boldsymbol{a}(r), \boldsymbol{b}(r), \boldsymbol{c}(r)\} \xrightarrow{\varphi} \boldsymbol{T} = \sum_{r=1}^{R} \boldsymbol{a}(r) \circ \boldsymbol{b}(r) \circ \boldsymbol{c}(r)$$

taking into account the presence of redundancies, the number of parameters in this parameterization is $M = R(N_1 + N_2 + N_3 - 2)$. In a canonical basis, $\boldsymbol{T}$ has the coordinate vector:

$$\sum_{r=1}^{R} \boldsymbol{a}(r) \otimes \boldsymbol{b}(r) \otimes \boldsymbol{c}(r)$$

where we may decide that $\boldsymbol{a}(r)$, $\boldsymbol{b}(r)$, and $\boldsymbol{c}(r)$ are row arrays of dimension N_1, N_2, and N_3, respectively, and $\otimes$ denotes the Kronecker product. Hence, after R iterations, the Jacobian of φ is the $R(N_1+N_2+N_3) \times N_1N_2N_3$ matrix:

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{I}_{N_1} & \otimes & \boldsymbol{b}(1) & \otimes & \boldsymbol{c}(1) \\ \boldsymbol{a}(1) & \otimes & \boldsymbol{I}_{N_2} & \otimes & \boldsymbol{c}(1) \\ \boldsymbol{a}(1) & \otimes & \boldsymbol{b}(1) & \otimes & \boldsymbol{I}_{N_3} \\ & & \vdots & & \\ \boldsymbol{I}_{N_1} & \otimes & \dots & \otimes & \dots \\ \dots & \otimes & \boldsymbol{I}_{N_2} & \otimes & \dots \\ \dots & \otimes & \dots & \otimes & \boldsymbol{I}_{N_3} \\ & & \vdots & & \\ \boldsymbol{I}_{N_1} & \otimes & \boldsymbol{b}(R) & \otimes & \boldsymbol{c}(R) \\ \boldsymbol{a}(R) & \otimes & \boldsymbol{I}_{N_2} & \otimes & \boldsymbol{c}(R) \\ \boldsymbol{a}(R) & \otimes & \boldsymbol{b}(R) & \otimes & \boldsymbol{I}_{N_3} \end{bmatrix} \tag{2}$$

The values of the generic rank obtained with this algorithm, called `rangj3(N1,N2,N3)`, or `rangj(N,L)` [5] for tensors of arbitrary order L and equal dimensions, are reported in tables 1, 2, and 3.

[5] The codes can be downloaded from `www.i3s.unice.fr/~pcomon`.

3.2 Jacobian for 3rd order asymmetric tensors with symmetric matrix slices

In this section, we consider tensors of size $N_2 \times N_2 \times N_3$, having symmetric $N_2 \times N_2$ matrix slices. We consider the mapping:

$$\{\boldsymbol{b}(r), \boldsymbol{c}(r)\} \longrightarrow \boldsymbol{T} = \sum_{r=1}^{R} \boldsymbol{b}(r) \circ \boldsymbol{b}(r) \circ \boldsymbol{c}(r).$$

Our code `rgindscal3(N2,N3)` implements the computation of the rank of the Jacobian below, when its size increases according to the algorithm described in section 3:

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{I}_{N_2} \otimes \boldsymbol{b}(1) \otimes \boldsymbol{c}(1) + \boldsymbol{b}(1) \otimes \boldsymbol{I}_{N_2} \otimes \boldsymbol{c}(1) \\ \boldsymbol{b}(1) \otimes \boldsymbol{b}(1) \otimes \boldsymbol{I}_{N_3} \\ \vdots \\ \boldsymbol{I}_{N_2} \otimes \boldsymbol{b}(R) \otimes \boldsymbol{c}(R) + \boldsymbol{b}(R) \otimes \boldsymbol{I}_{N_2} \otimes \boldsymbol{c}(R) \\ \boldsymbol{b}(R) \otimes \boldsymbol{b}(R) \otimes \boldsymbol{I}_{N_3} \end{bmatrix} \tag{3}$$

After R iterations, this matrix is of size $R(N_2 + N_3) \times N_2^2 N_3$. The number of parameters in this parameterization is $M = R(N_2 + N_3 - 1)$. Values of the generic rank are reported in table 4.

3.3 Jacobian for 3rd order double centered tensors with symmetric matrix slices

Now, take again $N_2 \times N_2 \times N_3$ tensors with symmetric $N_2 \times N_2$ matrix slices, but assume in addition that every row and column in the latter matrix slices are zero-mean. In order to achieve this, it is sufficient to generate vectors $\boldsymbol{b}(r)$ with zero mean [39]; in other words, only $N_2 - 1$ random numbers need to be drawn, the last entry of each vector $\boldsymbol{b}(r)$ being obtained via $b_{N_2} = -\sum_{n_2=1}^{N_2-1} b_{n_2}$. The Jacobian takes then the expression below:

$$\boldsymbol{J} = \begin{bmatrix} [\boldsymbol{I}_{N_2-1},\ -\mathbf{1}] \otimes \boldsymbol{b}(1) \otimes \boldsymbol{c}(1) + \boldsymbol{b}(1) \otimes [\boldsymbol{I}_{N_2-1},\ -\mathbf{1}] \otimes \boldsymbol{c}(1) \\ \boldsymbol{b}(1) \otimes \boldsymbol{b}(1) \otimes \boldsymbol{I}_{N_3} \\ \vdots \\ [\boldsymbol{I}_{N_2-1},\ -\mathbf{1}] \otimes \boldsymbol{b}(R) \otimes \boldsymbol{c}(R) + \boldsymbol{b}(R) \otimes [\boldsymbol{I}_{N_2-1},\ -\mathbf{1}] \otimes \boldsymbol{c}(R) \\ \boldsymbol{b}(R) \otimes \boldsymbol{b}(R) \otimes \boldsymbol{I}_{N_3} \end{bmatrix} \tag{4}$$

where $\mathbf{1}$ denotes a column of ones of size $N_2 - 1$. At the Rth iteration, this matrix is of size $R(N_2 + N_3 - 1) \times N_2^2 N_3$. The number of parameters in this parameterization is $M = R(N_2 + N_3 - 2)$. Table 5 reports some numerical values obtained with the code `rgindscal2z`.

3.4 Jacobian for 3rd order tensors with double centered matrix slices

The previous reasoning can be applied to $N_1 \times N_2 \times N_3$ tensors with no symmetry constraint and whose $N_1 \times N_2$ matrix slices have zero-mean rows and columns. As before, it is sufficient to generate vectors $\boldsymbol{a}(r)$ and $\boldsymbol{b}(r)$ with zero mean. The Jacobian is then equal to:

$$\boldsymbol{J} = \begin{bmatrix} [\boldsymbol{I}_{N_1-1},\ -\mathbf{1}] \otimes \boldsymbol{b}(1) \otimes \boldsymbol{c}(1) \\ \boldsymbol{a}(1) \otimes [\boldsymbol{I}_{N_2-1},\ -\mathbf{1}] \otimes \boldsymbol{c}(1) \\ \boldsymbol{a}(1) \otimes \boldsymbol{b}(1) \otimes \boldsymbol{I}_{N_3} \\ \vdots \\ [\boldsymbol{I}_{N_1-1},\ -\mathbf{1}] \otimes \boldsymbol{b}(R) \otimes \boldsymbol{c}(R) \\ \boldsymbol{a}(R) \otimes [\boldsymbol{I}_{N_2-1},\ -\mathbf{1}] \otimes \boldsymbol{c}(R) \\ \boldsymbol{a}(R) \otimes \boldsymbol{b}(R) \otimes \boldsymbol{I}_{N_3} \end{bmatrix} \tag{5}$$

At the Rth iteration, this matrix is of size $R(N_1 + N_2 + N_3 - 2) \times N_1 N_2 N_3$. The number of parameters in this parameterization is $M = R(N_1 + N_2 + N_3 - 3)$. The numerical values obtained with the code `rangj3z` are not reported, since we always have, for any triplet (N_1, N_2, N_3): `rangj3z(N1,N2,N3)`=`rangj3(N1-1,N2-1,N3)`. In other words, as far as the generic rank is concerned, centering in a given mode of dimension N_i yields the same effect as reducing the dimension to $N_i - 1$, which makes sense.

3.5 Jacobian for symmetric tensors

In the case of symmetric tensors of dimension N and order L, the mapping φ is defined from $\mathbb{K}^{NR}$ to the space of symmetric tensors [4], or equivalently to $\mathbb{K}^p$ with $p = \binom{N+L-1}{L}$, as:

$$\{\boldsymbol{a}(r) \in \mathbb{K}^N,\ 1 \leq r \leq R\} \xrightarrow{\varphi} \sum_{r=1}^{R} \boldsymbol{a}(r)^{\circ L}$$

where $\circ$ stands for the tensor (outer) product; once a basis is chosen, the tensor product may be expressed by a Kronecker product, yielding a similar

expression. In the case of order-3 tensors ($L = 3$) and after R iterations, the Jacobian of φ takes the following form, somewhat simpler than the previous cases:

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{I}_N \otimes \boldsymbol{a}(1) \otimes \boldsymbol{a}(1) + \boldsymbol{a}(1) \otimes \boldsymbol{I}_N \otimes \boldsymbol{a}(1) + \boldsymbol{a}(1) \otimes \boldsymbol{a}(1) \otimes \boldsymbol{I}_N \\ \vdots \\ \boldsymbol{I}_N \otimes \boldsymbol{a}(r) \otimes \boldsymbol{a}(r) + \boldsymbol{a}(r) \otimes \boldsymbol{I}_N \otimes \boldsymbol{a}(r) + \boldsymbol{a}(r) \otimes \boldsymbol{a}(r) \otimes \boldsymbol{I}_N \end{bmatrix} \tag{6}$$

This matrix is of size $RN \times N^3$, but we know that its rank cannot exceed $\binom{N+2}{3} = N(N+1)(N+2)/6$. The number of parameters in this parameterization is $M = RN$. Numerical values of the generic rank obtained with `rangjs(N,L)` are reported in table 6.

3.6 Jacobian for 4th order tensors with Hermitian symmetry

In this section, we consider fourth-order tensors of size $N \times N \times N \times N$ with Hermitian symmetry. We say that a fourth-order tensor $\boldsymbol{T}$ has Hermitian symmetry if it satisfies:

$$\boldsymbol{T}(i,j,k,l) = \boldsymbol{T}(k,j,i,l) = \boldsymbol{T}(i,l,k,j) = \boldsymbol{T}(j,i,l,k)^* \tag{7}$$

for all values of indices i, j, k and l. In particular, fourth-order complex circular cumulants have this Hermitian symmetry. The space of fourth-order tensors with Hermitian symmetry is denoted by $\mathcal{S}(N)$. We consider the following mapping from $\mathbb{C}^{NR}$ to $\mathcal{S}(N)$:

$$\{\boldsymbol{v}(r) \in \mathbb{C}^N,\ 1 \leq r \leq R\} \xrightarrow{\varphi_\epsilon} \sum_{r=1}^{R} \epsilon_r\, \boldsymbol{v}(r) \otimes \boldsymbol{v}(r)^* \otimes \boldsymbol{v}(r) \otimes \boldsymbol{v}(r)^*$$

in which $\epsilon_r = \pm 1$.

A symmetric real tensor has $N(N+1)(N+2)(N+3)/24$ degrees of freedom. However, in the complex case, a tensor enjoying Hermitian symmetries (7) has $N^2(N+1)^2/4$ distinct entries, of which some are strictly real. By counting the number of different real and imaginary parts of the tensor entries, we obtain that $\mathcal{S}(N)$ is a real vector space of dimension $K(N)$ [40, Annexe B]:

$$K(N) = 6\binom{N}{4} + 4\binom{N}{1}\binom{N-1}{2} + 3\binom{N}{2} + 2\binom{N}{1}\binom{N-1}{1} + \binom{N}{1}. \tag{8}$$

It is formed by the union of the images of φ_ϵ, for all $\epsilon = (\epsilon_1, \ldots, \epsilon_r)$. Note that ϵ_l does not affect the rank of the Jacobian. For clarity it will be omitted in the expressions below.

Function φ_ϵ is not analytic in $\mathbb{C}$. However, it is differentiable in $\mathbb{R}$. The Jacobian of φ_ϵ, interpreted as a mapping from $\mathbb{R}^{2NR}$ to $\mathcal{S}(N)$, now takes the following form:

$$\boldsymbol{J} = \begin{bmatrix} \mathrm{Re}(\gamma_{re}(\boldsymbol{v}(1))) & \mathrm{Im}(\gamma_{re}(\boldsymbol{v}(1))) \\ \mathrm{Re}(\gamma_{im}(\boldsymbol{v}(1))) & \mathrm{Im}(\gamma_{im}(\boldsymbol{v}(1))) \\ \vdots & \vdots \\ \mathrm{Re}(\gamma_{re}(\boldsymbol{v}(R))) & \mathrm{Im}(\gamma_{re}(\boldsymbol{v}(R))) \\ \mathrm{Re}(\gamma_{im}(\boldsymbol{v}(R))) & \mathrm{Im}(\gamma_{im}(\boldsymbol{v}(R))) \end{bmatrix} \tag{9}$$

with

$$\begin{aligned} \gamma_{re}(\boldsymbol{v}(r)) = {} & \boldsymbol{I}_N \otimes \boldsymbol{v}(r)^* \otimes \boldsymbol{v}(r) \otimes \boldsymbol{v}(r)^* + \boldsymbol{v}(r) \otimes \boldsymbol{I}_N \otimes \boldsymbol{v}(r) \otimes \boldsymbol{v}(r)^* \\ & + \boldsymbol{v}(r) \otimes \boldsymbol{v}(r)^* \otimes \boldsymbol{I}_N \otimes \boldsymbol{v}(r)^* + \boldsymbol{v}(r) \otimes \boldsymbol{v}(r)^* \otimes \boldsymbol{v}(r) \otimes \boldsymbol{I}_N \end{aligned}$$

and

$$\begin{aligned} \gamma_{im}(\boldsymbol{v}(r)) = {} & i\boldsymbol{I}_N \otimes \boldsymbol{v}(r)^* \otimes \boldsymbol{v}(r) \otimes \boldsymbol{v}(r)^* + \boldsymbol{v}(r) \otimes -i\boldsymbol{I}_N \otimes \boldsymbol{v}(r) \otimes \boldsymbol{v}(r)^* \\ & + \boldsymbol{v}(r) \otimes \boldsymbol{v}(r)^* \otimes i\boldsymbol{I}_N \otimes \boldsymbol{v}(r)^* + \boldsymbol{v}(r) \otimes \boldsymbol{v}(r)^* \otimes \boldsymbol{v}(r) \otimes -i\boldsymbol{I}_N. \end{aligned}$$

The matrix $\boldsymbol{J}$ is of size $2RN \times 2N^4$, but we know that its rank cannot exceed $K(N)$. The number of parameters in this parameterization is $M = 2RN$. Numerical values of the smallest typical rank obtained with `rangjh4(N)` are reported in table 7.

4 Numerical results

The available results on unconstrained, slicewise symmetric, and double centered arrays can be compared with the numerical values delivered by the computer codes.

Tensors with free entries. Table 1 reports typical ranks for 2-slice, 3-slice, and 4-slice arrays. All known typical rank values [29,31,32] are reported in plain face, and are compatible with the results from `rangj3`. Specifically, the smallest of the known typical rank values within a cell coincides throughout with the results from `rangj3`. For the yet unknown entries, we insert the results from `rangj3` in bold face. These bold face values represent the smallest typical rank in the real field, and the generic rank in the complex field.

	N_3	2				3			4	
N_1	N_2	2	3	4	5	3	4	5	4	5
2		2,3	3	4	4	3,4	4	5	4,5	5
3		3	3,4	4	5	5	5	5,6	**6**	**6**
4		4	4	4,5	5	5	**6**	**6**	**7**	**8**
5		4	5	5	5,6	5,6	**6**	**8**	**8**	**9**
6		4	6	6	6	6	**7**	**8**	**8**	**10**
7		4	6	7	7	7	**7**	**9**	**9**	**10**
8		4	6	8	8	8	8,9	**9**	**10**	**11**
9		4	6	8	9	9	9	**9**	**10**	**12**
10		4	6	8	10	9	10	10	**10**	**12**
11		4	6	8	10	9	11	11	**11**	**13**
12		4	6	8	10	9	12	12	12,13	**13**

Table 1
Typical ranks for 2-slice, 3-slice, and 4-slice unconstrained arrays. Values reported in bold correspond to smallest typical ranks computed numerically; values in plain font were known before. Values separated by commas are known typical ranks. In the complex field, the smallest value in a cell is generic.

The values reported in table 1 correspond to necessary conditions that ensure uniqueness of the CP, in the sense that a tensor having a rank larger than those generic values has infinitely many CPs in $\mathbb{C}$. Conversely, if the rank of a tensor is smaller than these generic values, then there is almost surely a finite number[6] of possible CPs up to a scale factor, in either $\mathbb{R}$ or $\mathbb{C}$. On the other hand, the bound given by Kruskal [24] corresponds to a sufficient condition ensuring uniqueness of the CP up to a permutation, and is always smaller than or equal to (but generally much smaller than) the generic rank. Other sufficient conditions have been derived in the literature under somewhat different assumptions, see e.g. [41] [42] for tensors enjoying Hermitean symmetries.

We report values of the smallest typical/generic rank of 3-way arrays with equal dimensions in table 2. The values shown in table 2 can also be compared to those obtained in the symmetric case (see table 6).

Now the algorithm can be run on tensors of order higher than 3. To make the presentation of the results readable, table 3 reports values of the generic

[6] This finite number of solutions may possibly be larger than the number of permutations. In such a case, the solution is not essentially unique.

N	2	3	4	5	6	7	8	9
$\bar{R}$	2	5	**7**	**10**	**14**	**19**	**24**	**30**

Table 2
Smallest typical rank $\bar{R}$ of unconstrained arrays of dimension $N \times N \times N$. Values reported in bold correspond to ranks computed numerically; values in plain font were known before. In the complex field, these values are generic.

L	N	2	3	4	5	6	7	8
3		2	5	**7**	**10**	**14**	**19**	**24**
4		**4**	**9**	**20**	**37**	**62**	**97**	

L	N	2	3	4	5	6	7	8
3		0	8	6	5	8	18	16
4		4	0	4	4	6	24	

Table 3
(top) Smallest typical rank $\bar{R}$ of unconstrained arrays of equal dimensions, N, and order L. In the complex field these values are generic. (bottom) Number F of remaining degrees of freedom; when $F = 0$, there are only a finite number of CP solutions.

rank obtained for asymmetric tensors with equal dimensions, N, and order L, with an algorithm referred to as `rangj(N,L)`. These results are consistent with those of [43]. We also indicate the dimensionality of the fiber of solutions. This number is simply defined as the difference:

$$F(N, L) = \bar{R}(N, L)\,(LN - L + 1) - N^L.$$

For those values of dimension and order for which $F = 0$, only a finite number of different CPs are possible.

Tensors with symmetric matrix slices. Having verified that `rangj3` and `rangj` work correctly throughout the cases where the generic/typical ranks are known, we next turn to the $N_1 \times N_2 \times N_2$ arrays with N_1 symmetric slices (Table 4). Again, known values coincide with numerical ones delivered by the code `rgindscal3`. We inserted results obtained from `rgindscal3` alone in bold face. As far as can be determined, all results are again in agreement with previously known values [33] [34].

Tensors with double centered symmetric matrix slices. When the matrix slices are symmetric and also row-wise (or column-wise, which is the same thing) zero-mean, the code `rgindscal2z` yielded the values reported in table 5. Note that the generic rank computed by `rgindscal2z(N2,N1)` is the same as that computed by `rgindscal3(N2-1,N1)`, at least according to the values explored in table 4. This shows that our method is easily adapted to handle

N_1	N_2	2	3	4	5
2		2,3	3,4	4,5	5,6
3		3	4	**6**	**7**
4		3	4,5	**6**	**8**
5		3	5,6	**7**	**9**
6		3	6	**7**	**9**
7		3	6	**7**	**10**
8		3	6	8	**10**
9		3	6	9,10	**11**
10		3	6	10	**11**

Table 4
Typical ranks for $N_1 \times N_2 \times N_2$ arrays, with $N_2 \times N_2$ symmetric slices. Values reported in bold correspond to smallest typical ranks computed numerically. Values separated by commas are known typical ranks. In the complex field, the smallest value in a cell is generic.

the special case of double centered matrices.

Tensors with double centered matrix slices without symmetry constraint. A similar observation holds also true when the centered matrix slices are not symmetric. We do not separately report typical rank values for the case of double-centered (non symmetric) slices. Instead, we verified that the values obtained numerically with centering coincided with the values obtained numerically for uncentered arrays: `rangj3z(N1,N2,N3)`=`rangj3(N1-1,N2-1,N3)`.

Symmetric tensors. In table 6 we report values obtained with 3-way or 4-way symmetric arrays, obtained with the code `rangjs`. Note that these results have been already reported in [4]. The dimensionality of the fiber of solutions is given by:

$$F(N,L) = \bar{R}\,N - \binom{N+L-1}{L}.$$

It is interesting to compare the ranks with those of the unsymmetric case, obviously larger, reported in table 3. In particular, one can observe that that the case $F = 0$ is again rarely met with generic arrays, but less rarely than in the non-symmetric case.

Fourth-order tensors with Hermitian symmetry. Finally, we report in table 7 values obtained for fourth-order tensors having Hermitian symmetry. These values were obtained with the code `rangj4h`.

N_1	N_2	2	3	4	5
2		1	2	3	4
3		1	3	4	**6**
4		1	3	**4**	**6**
5		1	3	**5**	**7**
6		1	3	6	**7**
7		1	3	6	**7**
8		1	3	6	8
9		1	3	6	9
10		1	3	6	10

Table 5
Smallest typical rank $\bar{R}$ for $N_1 \times N_2 \times N_2$ arrays, with $N_2 \times N_2$ symmetric slices having zero-mean columns. In the complex field, these values are generic.

L	N	2	3	4	5	6	7	8
3		2	4	5	8	10	12	15
4		3	6	10	15	21	30	42

L	N	2	3	4	5	6	7	8
3		0	2	0	5	4	0	0
4		1	3	5	5	0	0	6

Table 6
(top) Smallest typical ranks $\bar{R}$ of symmetric arrays of dimension N and order L. In the complex field, these values are generic. (bottom) Number F of remaining degrees of freedom; when $F = 0$, there are only a finite number of CP solutions.

N	2	3	4	5	6	7	8
$\bar{R}$	**4**	**9**	**16**	**25**	**41**	**61**	**87**

Table 7
Smallest typical rank $\bar{R}$ of fourth-order tensors with Hermitian symmetry of dimension $N \times N \times N \times N$. Values reported in bold correspond to ranks computed numerically.

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