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CLOSED-FORM EQUALIZATION

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ABSTRACT

Many blind or semi-blind equalizers are implemented with the help of iterative algorithms, and therefore may require long convergene times or suffer from local minima. Closed-form block blind equalizers are attractive, even if they are suboptimal, since they can serve to initialize them. But they can perform better in time-varying contexts, that is on very short data blocks. It is focused here mainly on PSK modulations, including MSK and QPSK. The proposed algorithms are based on known instantaneous properties of such modulations.[1]

1. INTRODUCTION

Blind equalization of a communication channel has been extensively studied and numerous recursive algorithms have been proposed. However finding reliable initializations appears as a very issue to improve on perfomances. On the other hand, the information given by the training sequence should allow to avoid local minima in semi-blind approaches.

Most popular approaches to blind equalization include the Constant Modulus (CM) property of the source signals [4] [17], or the kurtosis maximization [3] [13] together with fractional sampling at the receiver or induced cyclostationarity [1] [10] [14].

Another approach, that has recently received growing interest, is to use the discrete character of the digital communication sources. This information has been extensively used in decision directed equalizers [9, 11]. But Li [7, 8] showed that the use of the constellation as a criterion gives a consistent estimation of the sources and some recursive algorithms were proposed to achieve blind equalization [15, 12] or source separation [5].

In this paper, this latter approach is followed but we do look for closed-form solutions. An analytical solution has been already proposed in [16] for the separation of BPSK sources. However, compared to the existing works, our contribution is four-fold. We first show that the criterion proposed in [8] is MMSE equivalent for D-PSK sources and for high SNRs, then we propose an analytical solution for the SISO equalization of D-PSK sources in the blind and semi-blind contexts and investigate the asymptotical behavior of the proposed algorithm, we eventually give a simple way to avoid the drastic loss in performance when the equalizer length is wrongly determined. Moreover, the proposed solution can be easily applied to CM sources as well.

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Notations. Vectors are boldfaced and matrices are capitalized. The taps of a FIR filter $f(.)$ of length L will be stored in a column vector of size L, as $\mathbf{f}^T = [f(0)\cdots f(L-1)]$. A finite portion of length L of a time sequence $y(.)$ will be represented by a column vector of size L and denoted as :

$$\mathbf{y}(n;L)^T = [y(n)\, y(n-1)\,\cdots\, y(n-L+1)]$$

Next, given any complex vector $\mathbf{g}$ of size L, define $\mathbf{g}^{\oslash D}$ the column vector containing all the distinct order-D monomials built on the entries of $\mathbf{g}$ in an arbitrary fixed order. The monomials are weighted in order to preserve the Frobenius norm between the order-D rank-one tensor $G = \mathbf{g}\bullet\cdots\bullet\mathbf{g}$ (D times) and $\mathbf{g}^{\oslash D}$. Here, $\bullet$ stands for the outer product. For instance, if $L=3$ and $D=2$, then $\mathbf{g}^{\oslash 2} = \left[g_1^2, \sqrt{2}g_1g_2, \sqrt{2}g_1g_3, g_2^2, \sqrt{2}g_2g_3, g_3^2\right]^T$. Lastly, operators $\mathbf{vecs_D}\{.\}$ and $\mathbf{unvecs_D}\{.\}$ are defined as :

$$\mathbf{vecs_D}\{\mathbf{g}\bullet\cdots\bullet\mathbf{g}\} = \mathbf{g}^{\oslash q}$$
$$\mathbf{unvecs_D}\{\mathbf{g}^{\oslash D}\} = \mathbf{g}\bullet\cdots\bullet\mathbf{g}^T$$

2. PROBLEM STATEMENT

2.1. Context

Suppose we observe the output $y(n)$ of a channel $\mathbf{h}$ excited by a source signal $x(n)$ in presence of corrupting noise $w(n)$. Assuming linearity and local stationarity of the channel, the observation model in baseband takes the form below :

$$y(n) = \sum_{p=-\infty}^{\infty} h(p)\, x(n-p) + w(n) \qquad (1)$$

where $x(n)$ and $w(n)$ are assumed statistically independent. Our goal throughout this paper is to compute, in blind and semi-blind contexts, a Finite Impulse Response (FIR) filter, $f(n)$, of length L, so that its output $\hat{x}(n)$ approximates in some sense the source sequence $x(n)$, when input by $y(n)$:

$$\hat{x}(n) = \sum_{p=0}^{L-1} f(p)\, y(n-p) \qquad (2)$$

With the notations introduced in section 1, model (2) can be compactly written as : $\hat{x} = \mathbf{f}^T \mathbf{y}(n;L)$.

2.2. Derivation of the MMSE criterion

In a blind context and when the channel input is discrete, the equalizer estimator in the MMSE sense writes :

$$\mathbf{f} = \text{Arg} \min_{x(n)\in\mathcal{C},\mathbf{u}} E\left|x(n) - \mathbf{u}^T \mathbf{y}(n;L)\right|^2 \tag{3}$$

where $\mathcal{C}$ is the discrete support of the source, also referred to as the source constellation. If the noise is sufficiently weak, i.e. the eye is opened, the MMSE equalizer estimator (3) rewrites :

$$\mathbf{f} = \text{Arg} \min_{\mathbf{u}} E\left[\min_{x(n)\in\mathcal{C}} \left|x(n) - \mathbf{u}^T \mathbf{y}(n;L)\right|^2\right] \tag{4}$$

If the source constellation $\mathcal{C}$ is D-PSK modulated, then the min in (4) can be approximated by a product [5]. That is, if we observe N samples of sequence $y(n)$, (3) can be approximated by :

$$\mathbf{f} = \text{Arg} \min_{\mathbf{u}} \sum_{n=1}^{N} \left|\left(\mathbf{u}^T \mathbf{y}(n;L)\right)^D - 1\right|^2 \tag{5}$$

3. BLIND AND SEMI-BLIND EQUALIZATION

3.1. Blind equalization

Suppose the input source is D-PSK modulated, equation (5) shows that $\mathbf{f}$ is the Least Squares solution of the system of equations $\left(\mathbf{f}^T \mathbf{y}(i;L)\right)^D = 1$, $L \le i \le N$. Then applying $\mathbf{vecs_D}\{.\}$ to this system yields :

$$\begin{bmatrix} \mathbf{y}(L;L)^{\oslash D^T} \\ \vdots \\ \mathbf{y}(N;L)^{\oslash D^T} \end{bmatrix} \mathbf{f}^{\oslash D} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \tag{6}$$

This final system looks linear but the solution $\mathbf{f}^{\oslash D}$ must have a particular structure, as its elements verify some non-linear relations. Thus, ideally, we should use tools like Gröbner bases or Mc Aulay's theorem [2] to solve the above system. But the complexity of those methods rapidly increases with D and L.

Therefore, we prefer to look for $\mathbf{f}_{LS}$ in a suboptimal two-step method. First, we compute the unstructured Least squares solution $\mathbf{f}^{\oslash D}_{LS}$ by a mere pseudo-inversion of $Y(L:N;L) = \left[\mathbf{y}(L;L)^{\oslash D}, \cdots, \mathbf{y}(N;L)^{\oslash D}\right]^T$. Then, we force the structure by looking for the best rank-one approximation of $\mathbf{unvecs_D}\{\mathbf{f}^{\oslash D}_{LS}\}$.

3.2. Semi-blind equalization

Now, suppose that a part of the source signal is known, i.e. the training sequence. A possible way to use this knowledge, is to initialize $\mathbf{f}$ with the training sequence and then perform a gradient descent on the following combined optimization criterion :

$$\begin{aligned} \mathbf{f} = \quad & \text{Arg} \min_{\mathbf{u}} \left[\frac{\alpha}{N_1}\sum_{n=1}^{N_1} \left|\mathbf{u}^T \mathbf{y}(n;L) - x(n)\right|^2 \right. \\ & \left. + \frac{1-\alpha}{N-N_1}\sum_{n=N_1+1}^{N} \left|\left(\mathbf{u}^T \mathbf{y}(n;L)\right)^D - 1\right|^2\right] \end{aligned} \tag{7}$$

where parameter $\alpha \in [0,1]$ can be seen as a confidence rate in the known part of the signal.But we are looking for a closed-form solution.

Equation (7) is equivalent to solve the following system in the weighted LS sense :

$$\left\{\begin{aligned} \begin{bmatrix} \mathbf{y}(L;L) \\ \vdots \\ \mathbf{y}(N_1+L-1;L) \end{bmatrix} \mathbf{f} &= \begin{bmatrix} x(1) \\ \vdots \\ x(N_1) \end{bmatrix} \\ \begin{bmatrix} \mathbf{y}(N_1+L;L)^{\oslash D^T} \\ \vdots \\ \mathbf{y}(N;L)^{\oslash D^T} \end{bmatrix} \mathbf{f}^{\oslash D} &= \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \end{aligned}\right. \tag{8}$$

This system has two unknowns, $\mathbf{f}$ and $\mathbf{f}^{\oslash D}$, linked to each other. The suboptimal solution proposed consists of computing the dominant left singular vector of :

$$\left[\mathbf{unvecs_D}\{\mathbf{f}^{\oslash D}_{LS}\},\ \mathbf{f}_{LS}\right] W$$

where W is a weighting matrix, and $\mathbf{f}^{\oslash D}_{LS}$ and $\mathbf{f}_{LS}$ are LS solutions of the two systems in (8).

4. BLIND CASE ASYMPTOTICAL STUDY

We now concentrate on two issues. First, we focus on the invertibility conditions of matrix $Y(L:N;L)$ and then look whether the solution $\mathbf{f}^{\oslash D}_{LS}$ is close to the structured on.

4.1. Invertibility conditions

The pseudo-inversion of equation (6) writes :

$$Y(L:N;L)^\dagger Y(L:N;L)\,\mathbf{f}^{\oslash D} = Y(L:N;L)^\dagger \mathbf{d}$$

Now let A and $\mathbf{b}$ be defined by :

$$\begin{aligned} A &= \lim_{N\to\infty} Y(L:N;L)^\dagger Y(L:N;L) \\ \mathbf{b} &= \lim_{N\to\infty} Y(L:N;L)^\dagger \mathbf{d} \end{aligned}$$

Therefore, $A = [a_{ij}]$ is composed of estimations of some $2D$-order circular moments of $y(.)$ and the entries of $\mathbf{b}$ are estimations of some D-order noncircular moments of $y(.)$. Thus asymptotically, equation (6) rewrites : $A\,\mathbf{f}^{\oslash D} = \mathbf{b}$, where the a_{ij} are defined by :

$$a_{ij} = E\left[\prod_{d=1}^{D} y^*\left(n-\ell_i(d)\right) \prod_{d=1}^{D} y\left(n-\ell_j(d)\right)\right]$$

and $0 \leq \ell(d) \leq L$. The problem of interest is to find when A is rank deficient, that is, we have to investigate if linear relations between the a_{ij} exist for all i or for all j. It is quite obvious that in the presence of noise A is always full rank. So this study is carried out in the noiseless case.

4.1.1. MA channel

Suppose that the channel is MA and that the source is i.i.d (which is the case for D-PSK signals). Then the a_{ij} are given by :

$$a_{ij} = \sum_{m=1}^{M} h^*(m) \prod_{d=2}^{D} h^*\left(m + \ell_i(d) - \ell_i(1)\right) \prod_{d'=1}^{D} h\left(m + \ell_j(d') - \ell_i(1)\right)$$

Therefore, if the a_{ij} are linearly independent, then the taps of the channel h and their conjugate verify a non-trivial non-linear constraint. Hence, in the case of a MA channel, the matrix A is generically non-singular. This holds true when N is finite, because in this case matrix A_N is the sum of A plus the estimation noise.

4.1.2. AR channel

If we now suppose that the channel is AR, an exact finite length equalizer exists. Thus, in the noiseless case, if we try to find a too long equalizer, matrix A can be rank-deficient. Indeed, an AR channel introduces a linear dependency in the sequence $y(.)$:

$$b(0)y(n) = x(n) - \sum_{\ell=1}^{M} b(\ell)\, y(n-\ell)$$

where M is the length of the channel inverse. From this property, one can show that :

$$y(m) \left(\sum_{\ell=0}^{M} b(\ell) y(m-\ell) \right)^{D-1} = 1 - \sum_{\ell=1}^{M} b(\ell) y(m-\ell) \left(\sum_{\ell'=0}^{M} b(\ell') y(m-\ell') \right)^{D-1} \quad (9)$$

This shows that if $L \leq M$, A is generically invertible because $0 \leq \ell(d) \leq L$. But if $L > M$, (9) can be built with the entries of A for $m \in [n, \cdots, n-L+M]$ and the dimension of the null space of A is equal to $L-M$. It is easily shown that this property remains true for a finite number of taps N, as (9) is deterministic.

4.2. Structure issue

The previous analysis has shown that in the noiseless case and if the length of the equalizer is correct, the LS solution of (6) has the right structure. Indeed, since the matrix A is non-singular there is only one solution to (6) and since the correct equalizer satisfies (6), the LS solution is the good one and has the right structure.

We now study the effects of a bad estimation of the equalizer length on the LS solution without noise. In particular, we focus on the effects on its structure.

4.2.1. AR channel and $L > M$

We know from 4.1.2, that A has a null space whose dimension is $L - M$. In fact, the LS solution of (6) is the minimal norm linear combination of the symmetric vectorization of the rank-one tensors build on the $L - M + 1$ delayed versions of the exact channel inverse. Hence the LS solution will not have the right structure. A subspace based method was proposed in [6] to recover the structure in the case of the analytical CM algorithm. But, we propose here a simpler method that applies in the PSK and CM cases. If the first entry of a vector g of size L is zero, then the symmetric vectorization of the rank-one order-D tensors built on g has its L first entries equal to zero. Then, the M first entries of the LS solution of (6), $\mathbf{f}^{\oslash D}_{L}$, are equal up to a scalar factor to the M first entries of $\mathbf{f}^{\oslash D}_{M}$. The next $L - M$ entries of $\mathbf{f}^{\oslash D}_{L}$ are equal to zero. Therefore, in this case, a good estimation of $\mathbf{f}$ is given by the L first elements of $\mathbf{f}^{\oslash D}_{L}$.

4.2.2. MA or AR channel and $L < M$

Denote M the exact equalizer length, which can be infinite, and let L be the estimated length which satisfies $L < M$. The estimated and optimal equalizers are then given by $A_L \hat{\mathbf{f}}^{\oslash D}_{L} = \mathbf{d}_L$ and $A_M \mathbf{f}^{\oslash D}_{M} = \mathbf{d}_M$.

Using permutations, the latter equation rewrites :

$$\begin{bmatrix} A_L & Q \\ R & S \end{bmatrix} \begin{bmatrix} \hat{\mathbf{f}}^{\oslash D}_{L} \\ \delta \mathbf{f} \end{bmatrix} = \begin{bmatrix} \mathbf{d}_L \\ \delta \mathbf{d} \end{bmatrix}$$

so that $\left(\hat{\mathbf{f}}^{\oslash D}_{L} - \mathbf{f}^{\oslash D}_{L} \right) = A_L^{-1}\, Q\, \mathbf{d}_L$. This final equation shows that the structure issue depends on the channel itself, since A_L and Q are functions of $\mathbf{h}$ only. It also shows that the estimator of $\mathbf{f}^{\oslash D}_{L}$ is asymptotically biased. Nevertheless, simulations show that the L first entries $\mathbf{f}^{\oslash D}_{L}$ still give a good estimation of the equalizer.

5. SIMULATION RESULTS

In this section, we investigate the performances of our blind and semi-blind algorithms. Their behaviors are studied in terms of average Bit Error Rates for different SNRs, with either MSK or QPSK signals. The average performances are then compared to those of the CM algorithm, the Wiener and Zero-Forcing equalizers.

5.1. Blind equalization

Computer simulations have been run with QPSK signals. We used an AR channel for the need of comparison with the exact ZF equalizer. The poles of the AR channel were equal to : $0.354+0.354i$, $-.4i$, $-0.35-0.61i$ and $-0.8109+0.39i$. Figure 1 shows the average performance of our algorithm (dashed line), in the context of QPSK signals, for different SNRs. The results are then compared to the MMSE and ZF equalizers (solid lines) and to the analytical CM algorithm (dash-dotted line) [16] [6]. This shows that our algotithm behaves better than the CM algorithm for low SNRs.

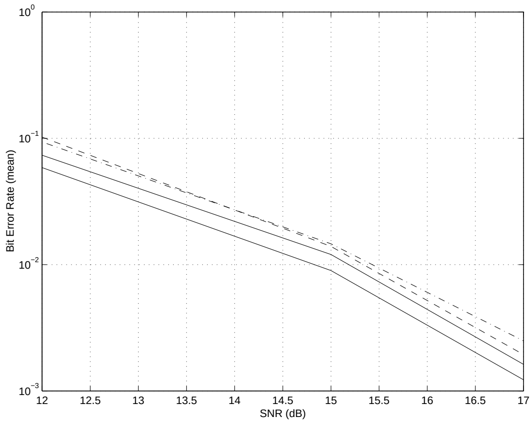

Figure 1: **Bit Error Rate as a function of SNR (averaging over 500 bursts of 200 samples)**

5.2. Semi-blind equalization

The semi-blind algorithm has been tested with the same AR channel and with MSK signals, as a function of α and at $11dB$ of SNR. Figure 2 shows the average BER (dashed line) for a training sequence of 20 samples. The results are compared to the ZF and MMSE equalizers (solid lines).

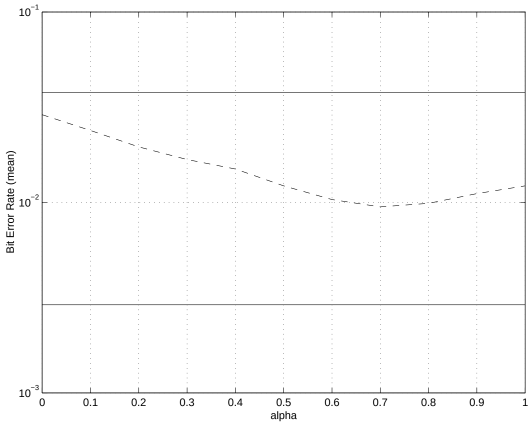

Figure 2: **Bit Error Rate as a function of α (averaging over 500 bursts of 200 samples)**

6. CONCLUSIONS

In this paper, closed-form solutions dedicated to blind and semi-blind equalization of PSK sources, including MSK and QPSK, are proposed. Simulation results show a good behavior of the different algorithms. In particular, we notice a gain in performance compared to the analytical CM algorithm when the size of the burst is small.

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